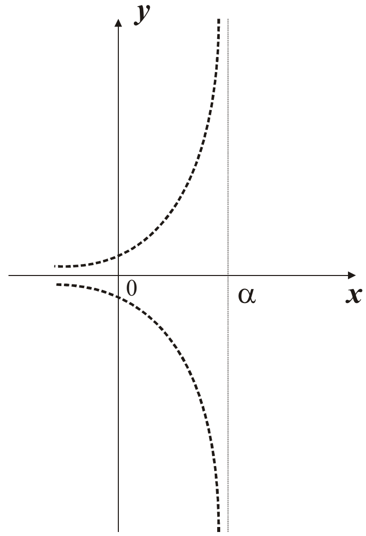
Further graph sketching

In this unit we are concerned with asymptotes of functions of the form $y = \frac{1}{(x-\alpha)}$.

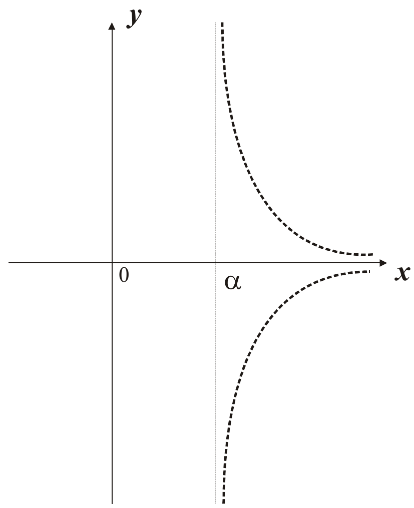
Graphs of this kind exhibit *singularities*; that is points where the function is not defined, and the curve is asymptotic to the vertical line $x = \alpha$. We are also concerned with sketching graphs of the form $y = |f(x)|$ and $y^2 = f(x)$. In both these cases you start by sketching the graph of $y = f(x)$; from that graph the other graphs may be deduced.

Singularities

The function $f(x) = \frac{1}{(x-\alpha)}$ cannot have a value at $x = \alpha$ since that would entail that the denominator became a zero. When $x = \alpha$, then $f(x) = \frac{1}{(x-\alpha)} = \frac{1}{(\alpha-\alpha)} = \frac{1}{0}$. It is not permitted to divide by zero, hence $f(x)$ cannot have a value at $x = \alpha$ and is undefined there. We call such points *singularities*. Around a singularity the graph of $y = f(x) = \frac{1}{(x-\alpha)}$ becomes asymptotic a vertical line $x = \alpha$. To determine which side of the *x*-axis the graph approaches the vertical asymptote, you test points around $x = \alpha$.



Near an asymptote as $x \to \alpha$ from below α, then does $f(x) \to +\infty$ or does $f(x) \to -\infty$? To find out test points just below $x = \alpha$



Near an asymptote as $x \to \alpha$ from above α, then does $f(x) \to +\infty$ or does $f(x) \to -\infty$? To find out test points just above $x = \alpha$

The form $y = |f(x)|$

The graph of $y = |f(x)|$ is identical to the graph of $y = f(x)$, except that any part of the graph of $y = f(x)$ that lies below the *x*-axis is reflected to lie above it. This clearly follows from the definition of the modulus.

$y = |f(x)| = f(x)$ if $y = f(x) > 0$
$y = |f(x)| = -f(x)$ if $y = f(x) < 0$

When the function takes the form $y^2 = f(x)$ then you begin by sketching the form $y = f(x)$. Then $y^2 = f(x)$ exhibits symmetry about the *x*-axis.

The form $y^2 = f(x)$

This exhibits symmetry about the *x*-axis. Since $\sqrt{y^2} = \pm y$, for every value of *y* that lies about the *x*-axis, there is a negative value below it. That is to determine the rough shape of this graph, note that if $|y| < 1$ then $|y^2| < |y|$, and if $|y| > 1$ then $|y^2| > |y|$

Example

The sketching of these curves is best illustrated by example.

<u>Question</u>

Make sketches of the graphs corresponding to each of the following functions

$(a) \qquad y = \frac{1}{x^2 - a^2}$

$(b) \qquad y = \left|\frac{1}{x^2 - a^2}\right|$

$(c) \qquad y^2 = \frac{1}{x^2 - a^2}$

where a is a positive constant. State the equation for each case of any asymptote(s) and the coordinates of any stationary points.

<u>Solution</u>

(a) We begin by skeching the graph of

$$y = f(x) = \frac{1}{x^2 - a^2} = \left(x^2 - a^2\right)^{-1}$$

$$\frac{dy}{dx} = -\left(x^2 - a^2\right)^{-2} \times 2x = -\frac{2x}{\left(x^2 - a^2\right)^2}$$

For turning points $\frac{dy}{dx} = 0$

Hence

$$-\frac{2x}{\left(x^2 - a^2\right)^2} = 0$$

$$x = 0$$

When $x = 0$ then $y = -\frac{1}{a^2}$

We need to test for the character of the stationary point.

Finding the second derivative could be tedious, so therefore we check what happens to $\frac{dy}{dx}$ around the point $x = 0$

When $x < 0,$ for example, when $x = -\frac{a}{2}$

$$\frac{dy}{dx} = -\frac{2x}{\left(\left(\frac{-a}{2}\right)^2 - a^2\right)^2} < 0$$

When $x > 0,$ for example, when $x = \frac{a}{2}$

$$\frac{dy}{dx} = -\frac{2x}{\left(\left(\frac{a}{2}\right)^2 - a^2\right)^2} > 0$$

So the point $\left(0, -\frac{1}{a^2}\right)$ is a max

We now need to find out about the asymptotes of this function..

When $x = a^2$ we have a singularity; that is, there are singularities at

$x = \pm a$

Thus, at these points there will be asymptotes which are parallel to the y-axis, with equations

$x = -a$

and

$x = a$

respectively.

As $x \to -a$ from the $x = 0$ side $y \to -\infty$ because $\left(0, -\frac{1}{a^2}\right)$ is a max, and there are no other turning points in the region. Similarly, $x \to +a$ from the $x = 0$ side then $y \to -\infty$

As $x \to -a$ from the $x < -a$ side, the demoninator of $y = \frac{1}{x^2 - a^2}$ is large and positive.

Hence as $x \to -a$ from the $x < -a$ side $y \to +\infty$.

Similarly, as $x \to a$ from the $x > a$ side $y \to +\infty$

As $x \to +\infty$, the demoninator of $y = \dfrac{1}{x^2 - a^2}$ becomes very large, hence

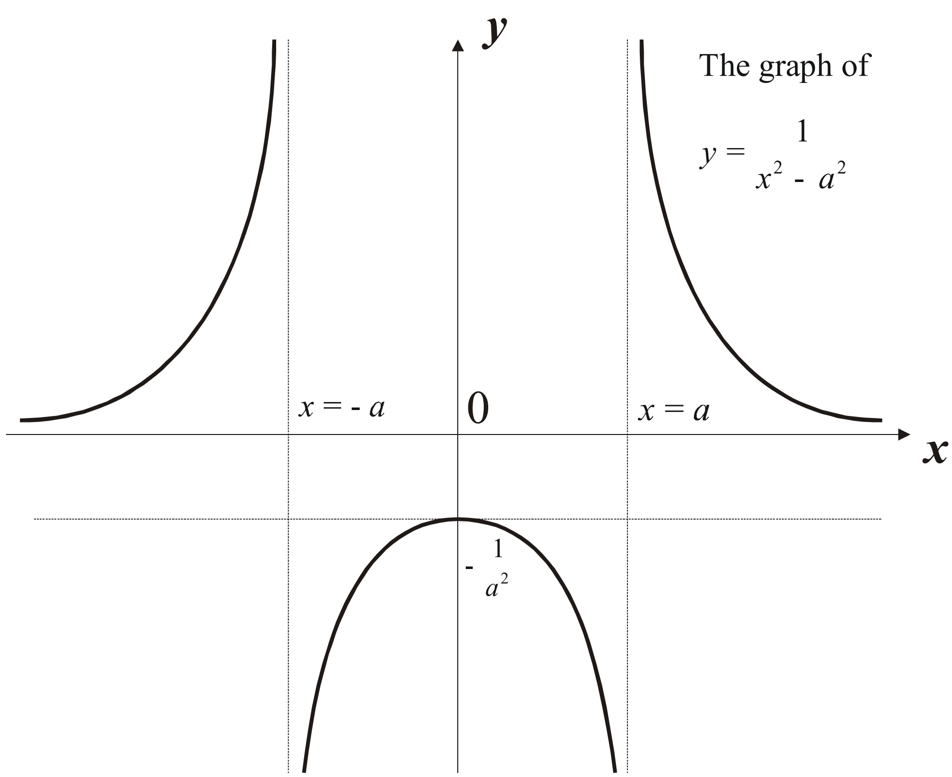
$y = \dfrac{1}{x^2 - a^2}$ becomes very small; that is as $x \to +\infty$, $y \to 0$

Similarly, $x \to -\infty$, $y \to 0$

In both cases the equation of the asymptote is

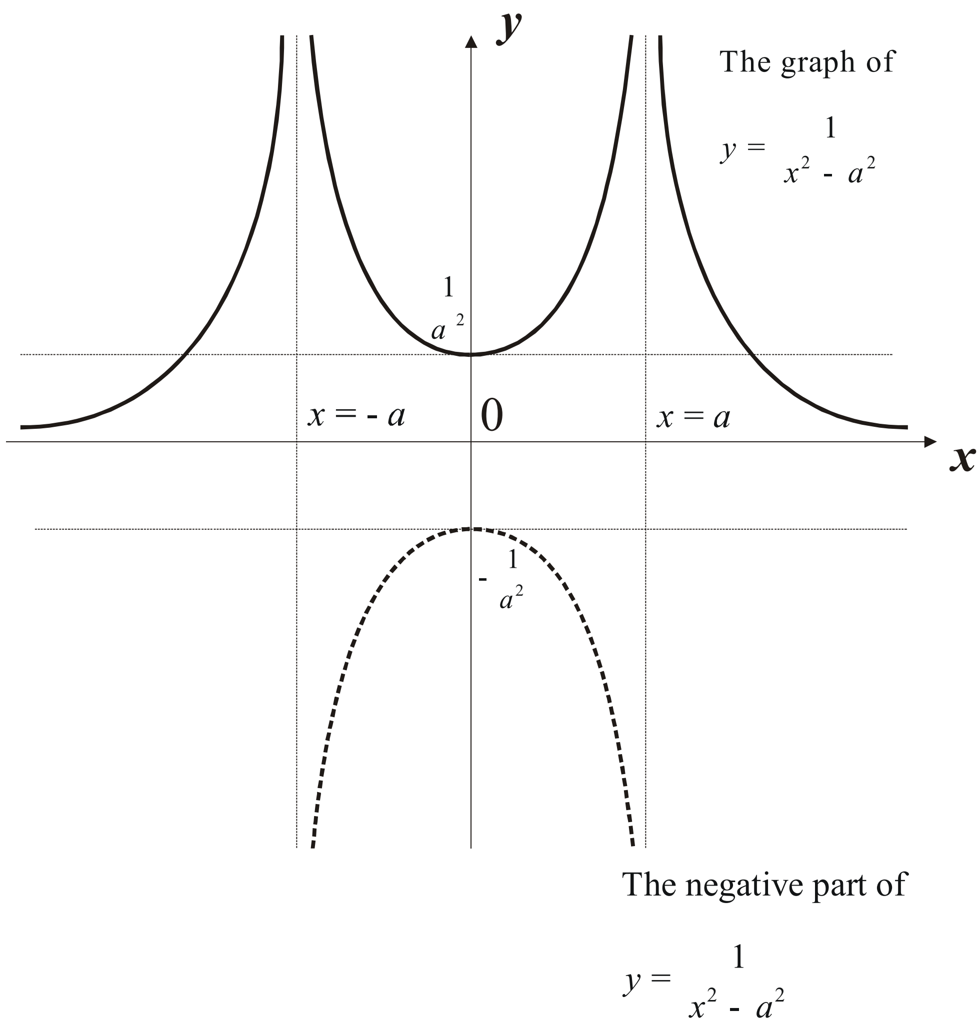
$y = 0$

We now have everything we need in order to sketch the graph.



$(b) \qquad y = \left| \dfrac{1}{x^2 - a^2} \right|$

This is derived from the graph of $y = \dfrac{1}{x^2 - a^2}$ by reflecting every part of $y = \dfrac{1}{x^2 - a^2}$ that lies below the x-axis, leaving the part that lies above the x-axis unchanged.



$(c) \qquad y^2 = \dfrac{1}{x^2 - a^2}$

This exhibits symmetry about the x-axis. Since $\sqrt{y^2} = \pm y$, for every value of y that lies about the x-axis, there is a negative value below it. That is

$$y = \pm\sqrt{\frac{1}{x^2 - a^2}}$$

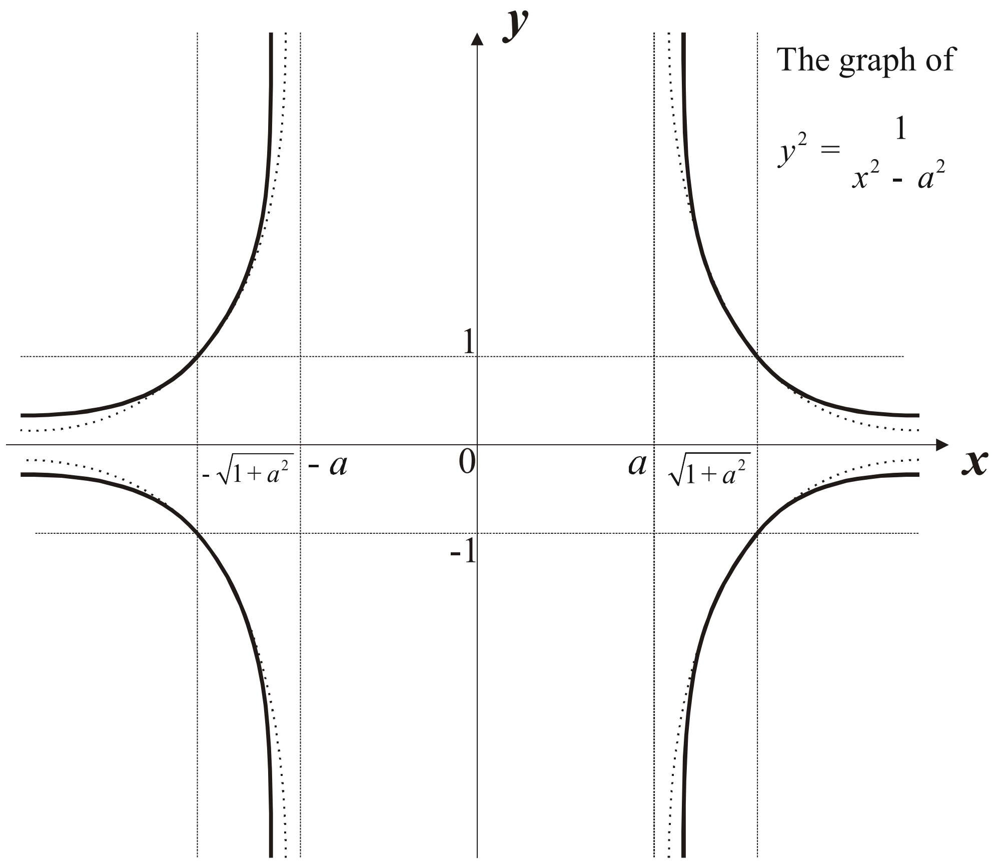
To determine the rough shape of this graph, note that if $|x| < 1$ then $\left|\frac{1}{x}\right| < \left|\frac{1}{x^2}\right|$, and if $|x| > 1$ then $\left|\frac{1}{x}\right| > \left|\frac{1}{x^2}\right|$. Hence $\left|\frac{1}{x}\right|$ lies below $\left|\frac{1}{x^2}\right|$ if $|x| < 1$, and above it if $|x| > 1$.

This means that $y^2 = \dfrac{1}{x^2 - a^2}$ lies below $y = \dfrac{1}{x^2 - a^2}$ if $x^2 - a^2 > 1$, which is to say, it will lie below $y = \dfrac{1}{x^2 - a^2}$ if $|x| > \sqrt{1 + a^2}$.

$y^2 = \dfrac{1}{x^2 - a^2}$ lies above $y = \dfrac{1}{x^2 - a^2}$ if $x^2 - a^2 < 1$, which is to say, it will lie above $y = \dfrac{1}{x^2 - a^2}$ if $|x| < \sqrt{1 + a^2}$.

The graph of $y^2 = \dfrac{1}{x^2 - a^2}$ does not exist if $\dfrac{1}{x^2 - a^2} < 0$ since there is no square root of a negative number. Hence the part of $y = \dfrac{1}{x^2 - a^2}$ that lies below the x-axis does not correspond to any part of $y^2 = \dfrac{1}{x^2 - a^2}$, and must be "deleted".

The graph of $y^2 = \dfrac{1}{x^2 - a^2}$ may now be sketched.



The dotted lines show the graph of

$y = \frac{1}{x^2 - a^2}$