Further loci of complex numbers

Harder sketching of complex loci

No new theory is introduced by the degree of algebraic manipulation required by the example that follows makes this a harder problem.

Example

Sketch the loci of the complex numbers that satisfy

$$\arg\left(\frac{z}{z+1-3i}\right)=\frac{\pi}{6}$$

Solution

We wish to sketch

$$\arg\left(\frac{z}{z+1-3i}\right)=\frac{\pi}{6}$$

To find the equation linking a and b let us begin by letting $z=a+bi$

Then

$$\arg\left(\frac{a+bi}{a+bi+1-3i}\right)=\frac{\pi}{6}$$

$$\arg\left(\frac{a+bi}{(a+1)+(b-3)i}\right)=\frac{\pi}{6}$$

This means that

$$\frac{\mathrm{Im}\left(\frac{a+bi}{(a+1)+(b-3)i}\right)}{\mathrm{Re}\left(\frac{a+bi}{(a+1)+(b-3)i}\right)}=\tan\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{3}$$

Before we can use this fact to solve the problem we must first remove the complex number from the bottom of the fraction by multiplying by its complex conjugate.

$$\frac{a+bi}{(a+1)+(b-3)i} = \frac{(a+bi)((a+1)-(b-3)i)}{((a+1)+(b-3)i)((a+1)-(b-3)i)}$$
$$= \frac{a(a+1)-a(b-3)i+b(a+1)i-b(b-3)i^2}{(a+1)^2-(b-3)^2 i^2}$$
$$= \frac{a^2+a-abi-3ai+abi+bi-b^2i^2+3bi^2}{a^2+2a+1-(b^2-6b+9)i^2}$$
$$= \frac{a^2+a-3ai+bi+b^2-b}{a^2+2a+1+(b^2-6b+9)}$$
$$= \frac{a^2+a+b^2-b+bi-3ai}{a^2+2a+1+b^2-6b+9}$$
$$= \frac{a^2+a+b^2-b+bi-3ai}{a^2+2a+b^2-6b+10}$$
$$= \frac{a^2+a+b^2-b}{a^2+2a+b^2-6b+10} + \frac{b-3a}{a^2+2a+b^2-6b+10}i$$

We can now use the relationship between real and imaginary parts to obtain the following.

$$b-3a = \frac{\sqrt{3}}{3}(a^2+a+b^2-b)$$
$$\frac{3}{\sqrt{3}}(b-3a) = a^2+a+b^2-b$$
$$\frac{3\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}(b-3a) = a^2+a+b^2-b$$
$$\frac{3\sqrt{3}}{3}(b-3a) = a^2+a+b^2-b$$
$$\sqrt{3}b-3\sqrt{3}a = a^2+a+b^2-b$$
$$0 = a^2+a+3\sqrt{3}a+b^2-b-\sqrt{3}b$$
$$0 = a^2+(1+3\sqrt{3})a+b^2-(1+\sqrt{3})b$$

We know that

$\left(a+\frac{1+3\sqrt{3}}{2}\right)^2 = a^2+(1+3\sqrt{3})a+\left(\frac{1+3\sqrt{3}}{2}\right)^2$ and

$\left(b-\frac{1+\sqrt{3}}{2}\right)^2 = b^2-(1+\sqrt{3})b+\left(\frac{1+\sqrt{3}}{2}\right)^2$, so

$\left(a+\frac{1+3\sqrt{3}}{2}\right)^2-\left(\frac{1+3\sqrt{3}}{2}\right)^2 = a^2+(1+3\sqrt{3})a$ and

$\left(b-\frac{1+\sqrt{3}}{2}\right)^2-\left(\frac{1+\sqrt{3}}{2}\right)^2 = b^2-(1+\sqrt{3})b$:

$$\left(a+\frac{1+3\sqrt{3}}{2}\right)^2-\left(\frac{1+3\sqrt{3}}{2}\right)^2+\left(b-\frac{1+\sqrt{3}}{2}\right)^2-\left(\frac{1+\sqrt{3}}{2}\right)^2=0$$

$$\left(a+\frac{1+3\sqrt{3}}{2}\right)^2+\left(b-\frac{1+\sqrt{3}}{2}\right)^2=\left(\frac{1+3\sqrt{3}}{2}\right)^2+\left(\frac{1+\sqrt{3}}{2}\right)^2$$

$$(a+3.098)^2+(b-1.366)^2=11.464\,(3.D.P.)$$

This is a circle with centre $(-3.098, 1.366)$ and radius 3.386 $(3.D.P.)$.

The locus of $\arg\left(\frac{z}{z+1-3i}\right)=\frac{\pi}{6}$