Linear simultaneous equations, matrices and Gaussian elimination

Prerequisites

You should be familiar with the addition and multiplication of matrices and be able to find the determinant of 2×2 and 3×3 matrices, as well as the inverse of a 2×2 matrix. You should understand the concept of linear dependence and know how to test for it by finding the determinant.

Linear dependence

The vector $\underline{\mathbf{a}}$ is said to be linearly dependent on the vectors $\underline{\mathbf{b}_1}, \underline{\mathbf{b}_2}, \underline{\mathbf{b}_3} \ldots \underline{\mathbf{b}_n}$

if and only, there exists numbers $\beta_1, \beta_2, \beta_3 \ldots \beta_n$

$\underline{\mathbf{a}} = \beta_1 \underline{\mathbf{b}_1} + \beta_2 \underline{\mathbf{b}_2} + \ldots. + \beta_n \underline{\mathbf{b}_n}$

The numbers could be of any type – real or complex.

For a square $n \times n$ matrix A

$\det A = 0$
A is singular
A^{-1} does not exist
A is linearly dependent
$\underline{\mathbf{a}} = \beta_1 \underline{\mathbf{b}_1} + \beta_2 \underline{\mathbf{b}_2} + \ldots. + \beta_n \underline{\mathbf{b}_n}$ can be solved for any rows or columns

} All these statements are equivalent

The purpose of this chapter is to apply your knowledge of matrices to the solution of problems involving simultaneous equations in two or three unknowns.

Systems of simultaneous equations

To each system of simultaneous equations there corresponds a matrix form called the *augmented matrix*. This is best shown through examples. To take one example, corresponding to the simultaneous equations

$$3x + 4y = 2$$
$$-2x - 3y = -1$$

the augmented matrix is

$$\left(\begin{array}{cc|c} 3 & 4 & 2 \\ -2 & -3 & -1 \end{array}\right).$$

This can be extended to any number of simultaneous equations; for example, for 3 equations in 3 unknowns. To

$$\begin{aligned} 2x + 2y - 3t &= -10 \\ x - y + 2t &= 8 \\ -x - 2y - t &= 3 \end{aligned}$$

the augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 1 & -1 & 2 & 8 \\ -1 & -2 & -1 & 3 \end{array}\right)$$

Most generally, a system M of equations in n unknowns

$$\begin{aligned} a_{11}\, x_1 \ + \ . \ . \ . \ a_{1n}\, x_n &= b_1 \\ &\vdots \\ a_{m1}\, x_1 + \ . \ . \ . \ a_{mn}\, x_n &= b_n \end{aligned}$$

has augmented matrix

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mm} & b_m \end{array}\right)$$

A solution to a set of simultaneous equations is a set of equations each in one variable. For example, the solution to

$$\begin{aligned} 3x + 4y &= 2 \\ -2x - 3y &= -1 \end{aligned}$$

is

$$\begin{aligned} x &= 2 \\ y &= -1 \end{aligned}$$

This solution could be written

$$\begin{aligned} x + 0y &= 2 \\ 0x + y &= -1 \end{aligned}$$

with augmented matrix

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

The solution to the second example

$$\begin{aligned} 2x + 2y - 3t &= -10 \\ x - y + 2t &= 8 \\ -x - 2y - t &= 3 \end{aligned}$$

is $x = 1,\ y = -3,\ t = 2$, to which the corresponding augment matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

We call matrices of this type, where the 'square' part of the augment matrix has the form of an identity matrix, the *row reduced* form. The solution to a system of simultaneous equations corresponds to the row reduced form of the augmented matrix. We, therefore, need operations on matrices that transform the augmented matrix of a system of simultaneous equations to the row reduced form. These row operations mimic the technique of elimination used to solve simultaneous equations. To show this, consider the solution, by elimination, of the first example.

(1)	$3x+4y=2$	
(2)	$-2x-3y=-1$	
$(1)\times(2)$	$6x+8y=4$	(3)
$(2)\times(3)$	$-6x-9y=-3$	(4)
$(3)+(4)$	$-y = 1$	
$\therefore$	$y=-1$	

Substituting $y=-1$ in (1)

$3x-4=2$

$3x=6$

$x=2$

In this the presence of the variables, *x* and *y*, are really non-essential; the process of elimination operates solely on the coefficients of the system of equations. Consequently, it makes sense to apply these operations to the augmented matrix alone. The augmented matrix is

$$\left(\begin{array}{cc|c} 3 & 4 & 2 \\ -2 & -3 & -1 \end{array}\right)$$

Applying these same row operations to this

$$\left(\begin{array}{cc|c} 6 & 8 & 4 \\ -6 & -9 & -3 \end{array}\right) \qquad \begin{array}{l} (1)\times 2 \\ (2)\times 3 \end{array}$$

$$\left(\begin{array}{cc|c} 3 & 4 & 2 \\ 0 & -1 & 1 \end{array}\right) \qquad \begin{array}{l} (1)\div 2 \\ (1)+(2) \end{array}$$

$$\left(\begin{array}{cc|c} 3 & 4 & 2 \\ 0 & -1 & 1 \end{array}\right) \qquad \begin{array}{l} \\ (2)\times 4 \end{array}$$

$$\left(\begin{array}{cc|c} 3 & 4 & 2 \\ 0 & -4 & 4 \end{array}\right) \qquad \begin{array}{l} (1)+(2) \\ (2)\div -4 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right) \qquad (1)\div 3$$

$\therefore\ x = 2$
$y = -1$

The row operations do to the rows of augmented matrix only what it is legitimate to do to the equations of a system of linear equations. To solve a system of linear equations we can add, subtract, multiply, divide and interchange the equations; similarly a row operation on a matrix is an operation of either

(1) adding a multiple of one row to another
(2) multiplying a row by a non-zero number
(3) interchanging two rows

The technique of row reduction is called *Gaussian elimination*. To further illustrate this, we use this technique to display the solution to the second example.

Example (1)

Solve

$$\begin{aligned} 2x + 2y - 3t &= -10 \\ x - y + 2t &= 8 \\ -x - 2y - t &= 3 \end{aligned}$$

Solution

The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 1 & -1 & 2 & 8 \\ -1 & -2 & -1 & 3 \end{array}\right)$$

The, using row operations

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 2 & -2 & 4 & 16 \\ -2 & -4 & -2 & 6 \end{array}\right) \qquad \begin{array}{l} (2)\times 2 \\ (3)\times 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 0 & 4 & -7 & 26 \\ 0 & -2 & -5 & -4 \end{array}\right) \qquad \begin{array}{l} (1)-(2) \\ (1)+(3) \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 0 & 4 & -7 & -26 \\ 0 & -4 & -10 & -8 \end{array}\right) \qquad (3)\times 2$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 0 & 4 & -7 & -26 \\ 0 & 0 & -17 & -34 \end{array}\right) \qquad (2)+(3)$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 0 & 4 & -7 & -26 \\ 0 & 0 & 1 & 2 \end{array}\right) \qquad (3)\div 17$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 0 & 4 & -7 & -26 \\ 0 & 0 & 7 & 14 \end{array}\right) \qquad (3)\times 7$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 0 & 4 & 0 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right) \qquad \begin{array}{l} (2)+(3) \\ (3)\div 7 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 0 & 2 & 0 & -6 \\ 0 & 0 & 3 & 6 \end{array}\right) \qquad \begin{array}{l} (2)\div 2 \\ (3)\times 3 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{array}\right) \qquad \begin{array}{l} (1)-(2)+(3) \\ (2)\div 2 \\ (3)\div 3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right) \qquad (1)\div 2$$

$\therefore\ x = 1,\ y = -3,\ z = 1$ is the solution

Echelon form

A matrix is said to be in *echelon form* if

(1) All zero rows are at the bottom of the matrix

(2) The leading entry of each nonzero row after the first occurs to the right of the leading entry of the previous row.

(3) The leading entry in any nonzero row is 1.

(4) All entries in the column below a leading 1 are zero.

Matrices in echelon form occur as a result of Guassian row reduction. However, according to this definition the only matrix in the previous example to be in echelon form is the last one

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

However, if we take the matrix

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 0 & 4 & -7 & -26 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

which occurred fifth line of the above and divide the second row by 4, then we get

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 0 & 1 & -\frac{7}{4} & -\frac{13}{2} \\ 0 & 0 & 1 & 2 \end{array}\right)$$

then this also would be in echelon form. The first point about echelon form is that when an augmented matrix is in echelon form either it is possible to read off the solutions directly from the matrix as in

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

or the final solution (if it exists) can be derived from it by further row operations. The echelon form may be thought of as a staging post on the way to solving a system of linear equations. When you are asked to solve a problem by *reduction to echelon form* you are being asked to use Gaussian row reduction to bring the augmented matrix to the form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{array}\right)$$

starting by getting the matrix into the form where each row has an increasing number of zero entries.

$$\left(\begin{array}{ccc|c} 1 & \dots & \dots & \dots \\ 0 & 1 & \dots & \dots \\ 0 & 0 & 1 & \dots \end{array}\right)$$

Here we have illustrated the definition of echelon form by 3×3 matrices, but the definition of echelon form applies to matrices of all sizes. However, at this level you would be unlikely to be asked to apply this. The following matrix is in echelon form.

$$\begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

But this matrix

$$\begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

is not.

Invertible matrices

A matrix A is said to be *invertible* if it has an inverse A^{-1}. You should already be aware that not all matrices are invertible. Those whose determinant is zero are not invertible and are said to be singular. When a matrix is invertible, the inverse A^{-1} can also be found by Gaussian row reduction, as the following example illustrates.

Example (2)

Use reduction to echelon form to find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution

Two matrices, A and A^{-1}, are inverses of each other when

$$AA^{-1} = \mathbf{I}$$

In this case

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We write this equation in a form of augmented matrix thus

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Now let us apply row operations on both halves of this augmented form with the rule that whatever is done to one half must also be done to the other.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \qquad (2)-(1)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 1 & -1 \end{array}\right) \qquad (2)-(3)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right) \qquad (3) \div -2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right) \qquad (2)+(3)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right) \qquad (1)-(3)$$

From this the inverse is

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

You can check this if you like by showing $AA^{-1} = A^{-1}A = \mathbf{I}$

Now consider the system of simultaneous equations

$$x + z = 1$$
$$x + y = 2$$
$$y + z = 3$$

This has the form

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

To make this absolutely clear, we can write $A\mathbf{x} = \mathbf{b}$ as

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

then multiplying out the matrices and uncoupling them gives us back the original set of simultaneous equations

$$x + z = 1$$
$$x + y = 2$$
$$y + z = 3$$

The point is that from

$$A\mathbf{x} = \mathbf{b}$$

we obtain the solution to the system as

$$\mathbf{x} = A^{-1}\mathbf{b}$$

This makes it clear that the equation $A\mathbf{x} = \mathbf{b}$ only has a solution if A is invertible, that is, has an inverse.

Example (3)

Using the result from example (2) solve

$$x + z = 1$$
$$x + y = 2$$
$$y + z = 3$$

Solution

Putting

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

as above, then

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{x} &= A^{-1}\,\mathbf{b} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \end{aligned}$$

The solution is $x = 0,\ y = 2,\ z = 1$

Now let us explore what happens when the matrix A in the equation $A\mathbf{x} = \mathbf{b}$ is not invertible.

Example (4)

The matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

What happens when you try to solve the simultaneous equations

$$\begin{aligned} x - y &= 1 \\ y - z &= 2 \\ x - z &= 3 \end{aligned}$$

by the method of reduction to echelon form? !f the equations were

$$\begin{aligned} x - y &= 1 \\ y - z &= 2 \\ x - z &= 1 \end{aligned}$$

how would this affect the result?

Solution

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & -1 & 3 \end{array}\right)$$

Applying row reduction we get

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right) \qquad (3)-(1)$$

This is already in echelon form, but we see that the second and third rows are identical, and we cannot proceed any further. Since A is singular any one row is a linear combination of the other two, and this process of reduction just confirms that by showing that in effect we do not have three equations in three unknowns, but actually only two. Since one equation is the linear combination of the two others, it is strictly redundant. However, the system is consistent and does have a solution. The solution is not a point but a line. Putting $z = t$ then

$$y - t = 2$$
$$y = 2 + t$$
$$x - y = 1$$
$$x = 1 + y = 1 + (2 + t) = 3 + t$$

So the solution is the line given by the vector

$$\mathbf{r} = \begin{pmatrix} t+3 \\ t+2 \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

Now if instead the system was

$$x - y = 1$$
$$y - z = 2$$
$$x - z = 1$$

Then the echelon form would be

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 0 \end{array}\right) \qquad (3)-(1)$$

Now we have two equations that are inconsistent

$$y - z = 2$$
$$y - z = 0$$

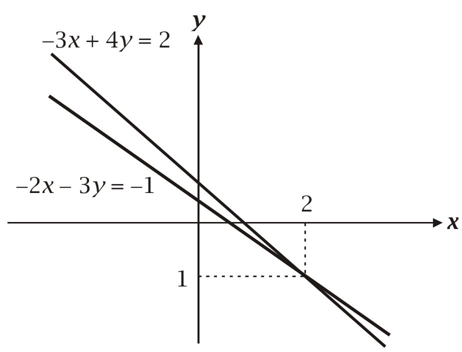
So this system cannot have a solution.

Graphical interpretation

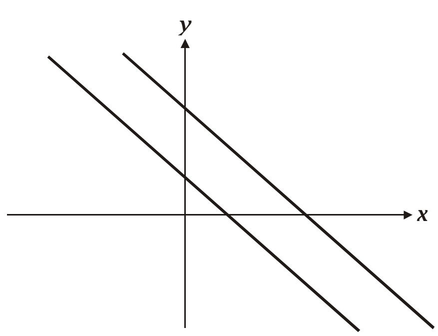
Every system of simultaneous equations can be interpreted geometrically. For example, corresponding to the equations

$3x + 4y = 2$
$-2x - 3y = -1$

We can view the solution, $x = 2$ $y = -1$ as the point of intersection of the lines



The system of simultaneous equations will have a unique solution if the graphical representation shows the two lines crossing over to give a point of intersection. However, the graph makes it clear that this need not always be the case. When the two lines corresponding to two equations are parallel there is no solution to the set of simultaneous linear equations:-



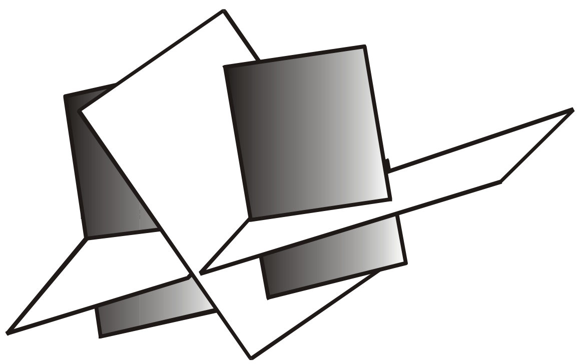
Parallel lines do not give a point of intersection. In two dimensions two lines are parallel when one is a multiple of another.

Thus

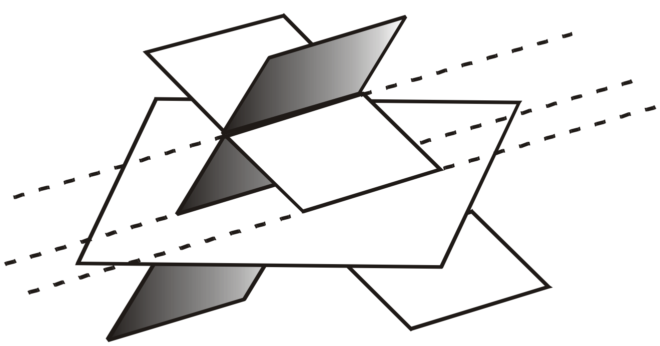
$3x + 4y = 2$ (1)
$6x + 8y = 4$ (2)

are parallel since $(2) = (1) \times 2$

This shows that the system is linearly dependent. Only linearly independent systems of simultaneous equations have unique solutions. This also applies in three and in *n* dimensions. We should also explore the visual interpretations of what happens in three dimensions when a system of three equations has and has not a unique solution. Each equation in the system defines a plane in 3-dimensional space. Thus, for there to be a unique solution, all three planes must intersect at a unique point



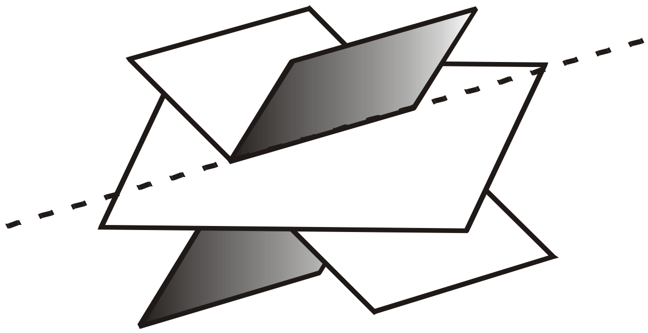
The diagram makes it clear that having a unique point of intersection is only one possibility. One plane may be parallel to another. Alternatively, each plane may intersect each other to give a line, but the lines may not intersect uniquely



This corresponds to the case in 3-dimensions where the system of equations is inconsistent. We saw in example (4) that the system

$x - y = 1$
$y - z = 2$
$x - z = 1$

was inconsistent. Each pair of equations defines an intersection of two planes giving rise to a line. But the three lines created do not coincide and the system as a whole is inconsistent. On the other hand, three equations might intersect in such a way as to define one line



This is the situation where the system of equations is linearly dependent, so one equation at least is redundant; nonetheless the system is consistent. We saw in example (4) that the system

$$x - y = 1$$
$$y - z = 2$$
$$x - z = 3$$

was of this type.