Geometric Progressions

Definition of a geometric progression

A *sequence* is any string of numbers in a given order. In a *geometric sequence* or *geometric progression* (equivalent expressions) each number in the sequence is related to the next by a particular rule.

Example (1)

Consider the following sequence.

1 2 4 8 16 32 64

How is each successive term generated from the one that precedes it?

Solution

$$1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{\times 2} 32 \xrightarrow{\times 2} 64$$

At each stage we are multiplying the preceding term by 2. Thus, this sequence of numbers is defined by

The first term: in this example = 1

The common ratio between the terms: in this example = 2

This is an example of a *geometric progression.* A *geometric progression* is a sequence in which each successive term is found from the preceding one by multiplying it by a fixed number, called the ratio. This means the *ratio* of any two successive terms is constant. Let $u_1, u_2, u_3, \ldots, u_n$ represent the first, second, third and nth term of any sequence. With first term a and ratio r a geometric progression has the general form

$$\begin{aligned} u_1 &= a \\ u_2 &= ar \\ u_3 &= ar^2 \\ &\vdots \\ u_n &= ar^{n-1} \end{aligned}$$

Example (2)

Find the eighth term of the geometric progression

2, –6, 18, –54, …

Solution

The first term is

$a = 2$

We need to find the ratio, *r*. The second term is

$u_2 = ar = -6$

So if we divide the first term by the second

$r = \frac{ar}{a} = \frac{-6}{2} = -3$

Now that we know the ratio, we can find the eighth term.

$u_8 = ar^7 = 2 \times (-3)^7 = -4374$

The sum of a geometric progression

The sum of a geometric progression of n terms is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$

Later we will prove this formula, but for the present we will use it.

Example (3)

(*a*) The 1st and the 5th terms of a geometric progression are 5 and 0.128 respectively. Find the common ratio and the 7th term .

(*b*) Find the sum of the first 10 terms, giving your answer to 3 significant figures.

Solution

$u_0 = a = 5$

$u_5 = ar^4 = 0.128$

$r^4 = \frac{ar^4}{a} = \frac{0.128}{5} = 0.0256$

$r = \sqrt[4]{0.0256} = 0.4$

This gives us the ratio, so the 7th term is

$u_7 = ar^6 = 5 \times 0.4^6 = 0.02048$

Substituting $a = 5, r = 0.4, n = 10$ into $S_n = \frac{a(1-r^n)}{1-r}$ we get

$$
\begin{aligned}
S_n &= \frac{a\left(1-r^n\right)}{1-r} \\
&= \frac{5\left(1-0.4^{10}\right)}{1-0.4} \\
&= \frac{5(1-0.0001048576)}{0.6} \\
&= 8.33245952 \\
&= 8.33 \ (3.\text{s.f.})
\end{aligned}
$$

Geometric progressions can be *divergent* or *convergent* (or constant, if the ratio is 1). The term divergent means that the successive terms of the series get larger and larger. The series in example (3) is convergent, meaning that each successive term is smaller than the one preceding it.

$$
\begin{aligned}
u_0 &= a = 5 \\
u_1 &= ar = 5 \times 0.4 = 2 \\
u_2 &= ar^2 = 2 \times 0.4 = 0.8 \\
u_3 &= ar^3 = 0.8 \times 0.4 = 0.32 \\
u_4 &= ar^4 = 0.32 \times 0.4 = 0.128
\end{aligned}
$$

So the terms are clearly getting smaller and smaller. If a series is divergent then it cannot possibly have a sum to infinity because the sum is getting larger and larger with each successive term in the series. However, it turns out that when a geometric progression is convergent then the sum of a geometric progression is also convergent. This means that the sum gets closer and closer to a certain fixed number, which is called the *limit* of the series of sums. This limit is given by the formula

$$S_\infty = \frac{a}{1-r} \quad \text{if and only if } |r| < 1$$

This states that the *sum of a geometric series to infinity* is $\frac{a}{1-r}$, if and only if the modulus of the common ratio, *r*, is less than 1. (The expression "if and only if" in this context it means (*a*) if the geometric series is convergent then the modulus of the common ratio is less than 1, and (*b*) if the modulus of the common ratio is less than 1, then the geometric series is convergent.)

Example (3) continued

(*c*) The geometric progression in example (3) had first term 5 and common ratio 0.4. Find its sum to infinity.

Solution

(*c*) On substituting $a = 5, r = 0.4$ into $\frac{a}{1-r}$

$$S_\infty = \frac{a}{1-r} = \frac{5}{0.6} = 8.3\dot{3}$$

Problems based on geometric progressions

Problems can be set requiring you to find an unknown quantity. Consider this example.

Example (4)

The sum to infinity of a geometric series is 3.75. The common ratio is positive and the sum of the first two terms is 3.6. Find the first term, the common ratio and the sum of the first 10 terms.

Solution

The sum to infinity of a geometric series is 3.75 translates to

$$\frac{a}{1-r}=3.75$$

The common ratio is positive and the sum of the first two terms is 3.6 translates to

$$a+ar=3.6$$
$$a(1+r)=3.6$$
$$a=\frac{3.6}{1+r}$$

Substituting into the first equation gives

$$\frac{\left(\frac{3.6}{1+r}\right)}{1-r}=3.75$$
$$\frac{3.6}{(1+r)(1-r)}=3.75$$
$$3.75(1+r)(1-r)=3.6$$
$$3.75(1-r^2)=3.6$$
$$3.75-3.75r^2=3.6$$
$$3.75r^2=0.15$$
$$r^2=0.04$$
$$r=\pm 0.2$$

We are told that r is positive so the solution is $r=0.2$

The first term is given by

$$a=\frac{3.6}{1+r}=\frac{3.6}{1.2}=3$$

The sum of the first10 terms is

$$\begin{aligned}S_n&=\frac{a(1-r^n)}{1-r}\\&=\frac{3(1-0.2^{10})}{1-0.2}\\&=3.749999...=3.75\ \text{(3.s.f.)}\end{aligned}$$

Problems on geometric progressions can involve you in manipulation of logarithms. The following problem will lead to an index equation that will require logarithms to solve.

Example (5)

The first term of a geometric progression is 5, and its ratio is 4. Find the number of terms of the sequence so that the sum is as great as possible but less than 10,000.

Solution

The strategy in answering this question is to substitute the first term and common ratio into the equation for the sum of the geometric progression.

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

Then solve this equation for n, the number of terms in the progression. This will require logarithms. This will give n as a real number. Then the answer will be the next integer down from n. So on substituting into $S_n = \frac{a\left(1-r^n\right)}{1-r}$ we get

$$10{,}000 = \frac{5\left(1-4^n\right)}{1-4}$$

$$-30000 = 5 - 5 \times 4^n$$

$$4^n = 6001$$

$$\log 4^n = \log 6001$$

$$n = \frac{\log 6001}{\log 4} = 6.27\ldots$$

$$\therefore \text{ number of terms} = 6$$

Proofs of the formulae for the sum of a geometric progression

To prove that the sum of a geometric progression of n terms is given by

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

where a is the first term and r the common ratio of the progression.

Proof

The sum of a geometric progression of n terms is given by

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

Multiplying this equation on both left and right hand sides by r gives

$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$

Subtracting the second equation from the first gives

$S_n - rS_n = a - ar^n$

The middle terms cancel out in pairs. So

$S_n(1-r) = a(1-r^n)$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

Convergent and divergent sums

Successive sums of a geometric progression define a *series*. This series can be divergent or convergent. For a geometric progression when $|r| < 1$ the series of sums is convergent. Looking at the formula for the sum

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

we see that as $n \to \infty$, $r^n \to 0$ (which is read, "As n tends to infinity, r^n tends to zero").

This in turn means that as $n \to \infty$, $(1-r^n) \to 1$ ("As n tends to infinity, $1-r^n$ tends to 1".). Hence

$$S_\infty = \frac{a}{1-r} \text{ if } |r| < 1$$

If $|r| > 1$ then the series of successive sums is divergent. In the expression $S_n = \frac{a(1-r^n)}{(1-r)}$ the r^n term gets bigger and bigger as $n \to \infty$. If $|r| = 1$ the expression $S_n = \frac{a(1-r^n)}{(1-r)}$ is also divergent. Each term in the progression is the same number, and the addition of the same number to itself n times creates an ever increasing sum as $n \to \infty$. Hence

$$S_\infty = \frac{a}{1-r} \text{ if and only if } |r| < 1$$

meaning

$S_\infty = \frac{a}{1-r}$ if $|r| < 1$ and $|r| < 1$ if $S_\infty = \frac{a}{1-r}$.