Geometrical uses of complex numbers

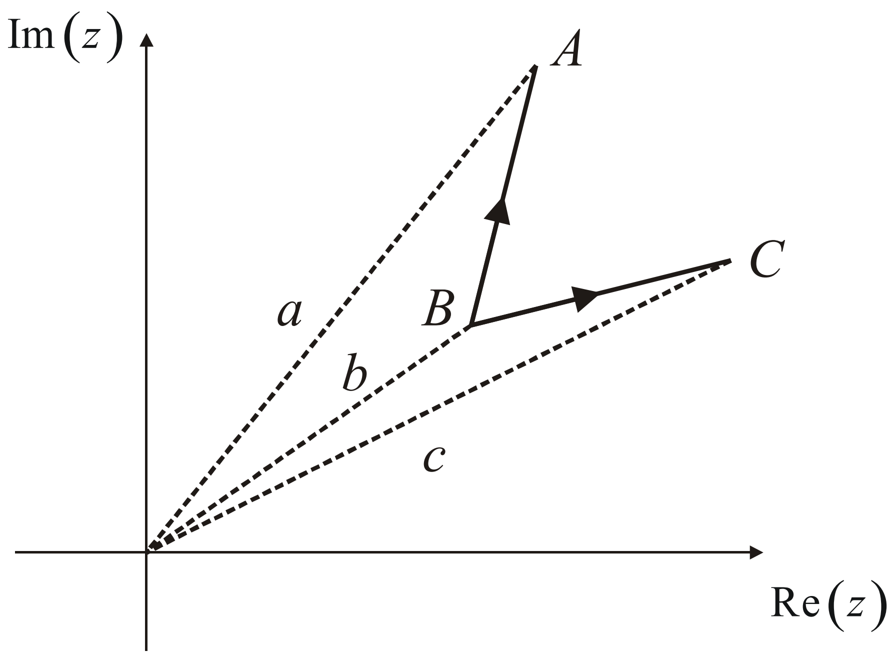
Recall if w and z are complex numbers then

$$\arg(wz) = \arg(w) + \arg(z)$$
$$|wz| = |w||z|$$

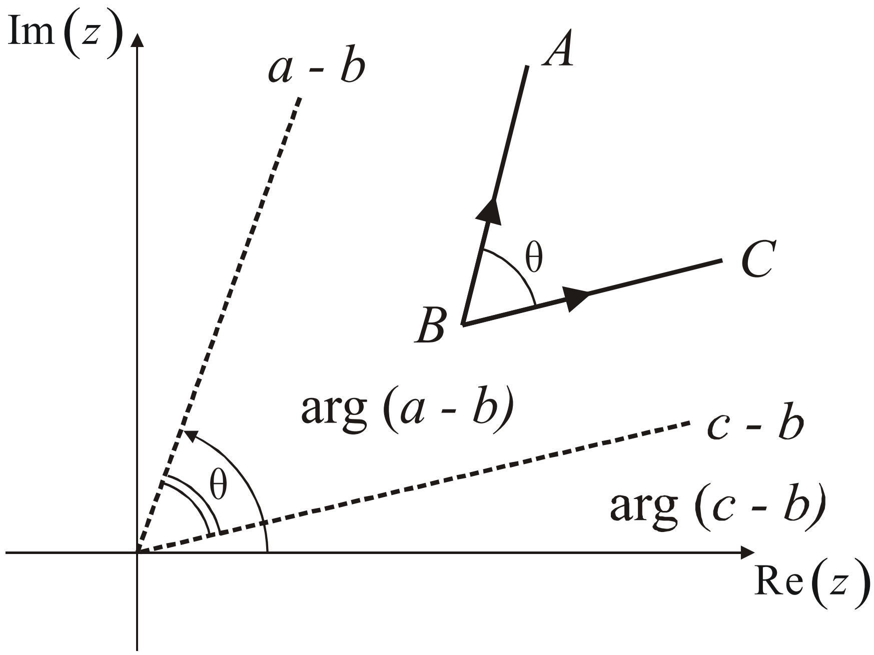
In other words, when multiplying complex numbers you add the arguments and multiply the
moduli.
This properly can be used to establish results about geometrical figures.

Firstly, we shall show that if A, B, C are three points in the Argand plane corresponding to complex numbers a, b, c then

$$\arg\left(\frac{a-b}{c-b}\right) = angleABC$$



The vector $\overrightarrow{BA}$ is represented by the complex number $a-b$. The vector $\overrightarrow{BC}$ is represented by the complex number $c-d$.



Hence, the angle ϕ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ is equal to the angle between $a-b$ and $c-b$.
That is

angle $ABC = \phi = \arg(a-b) - \arg(c-b)$
there, the direction of $(a-b)$ must be ϕ ahead of the direction $(c-b)$.

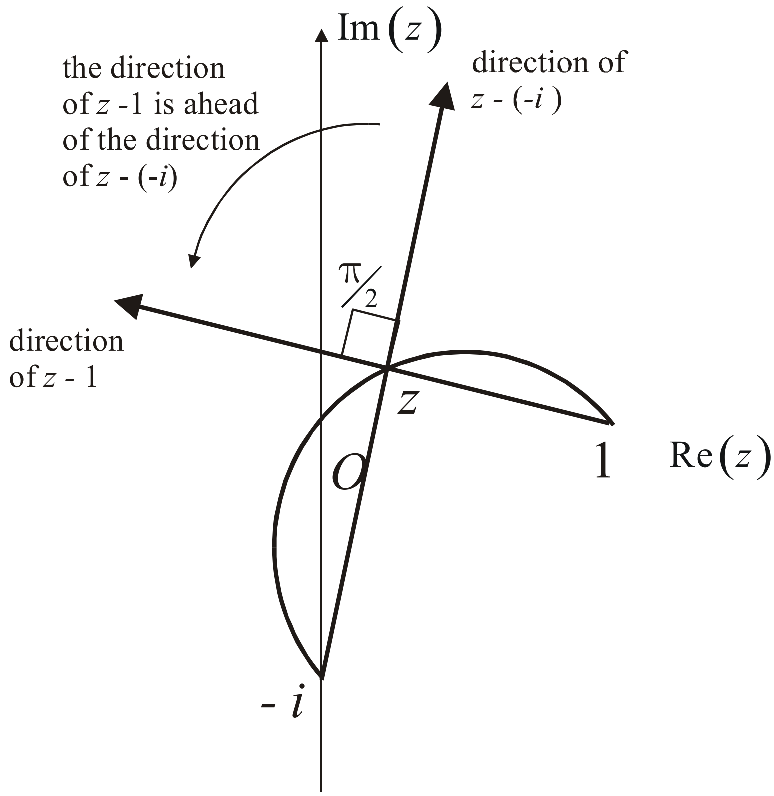
Example (1)

Determine the locus of the points z such that $\arg\left(\frac{z-l}{z+i}\right) = \pi/2$

Solution

$$\arg\left(\frac{z-l}{z+i}\right) = \arg\left(\frac{z-l}{z-(-i)}\right)$$

The complex number $z-l$ represents the line joining l to z; the complex number $z-(-i)$ represents the line joining $-i$ to t. The condition $\arg\left(\frac{z-l}{z-i}\right) = \pi/2$ means that the angle between these two lines must be $\pi/2$ (a right angle). Also the direction of $z-i$ must be ahead (by $\pi/2$) of the direction $z-i$.



<u>Example (2)</u>

Determine the locus of the points z such that $\arg\left(\frac{z}{z-2-i}\right)=\frac{\pi}{4}$ and

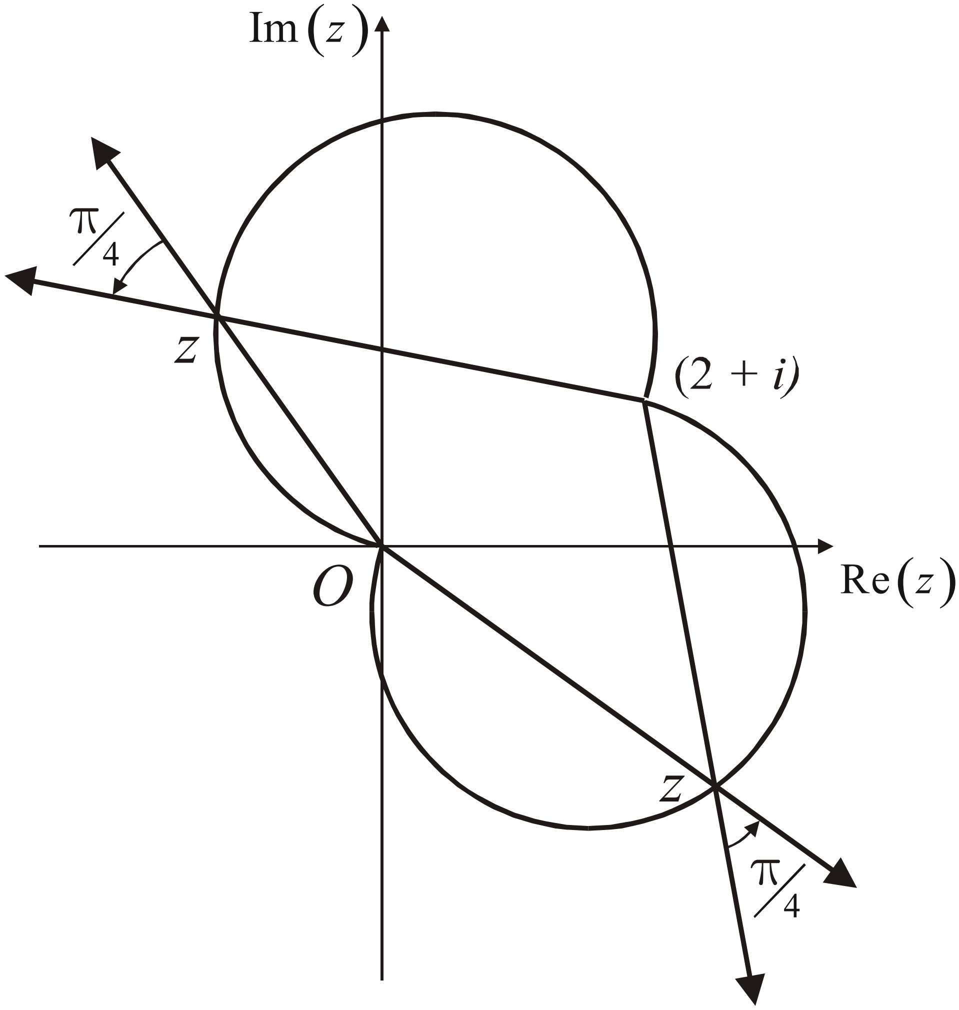
$$\arg\left(\frac{z}{z-2-i}\right)=\frac{\pi}{4}$$

<u>Solution</u>

$$\arg\left(\frac{z}{z-2-i}\right)=\arg\left(\frac{z-o}{z-(2+i)}\right)$$

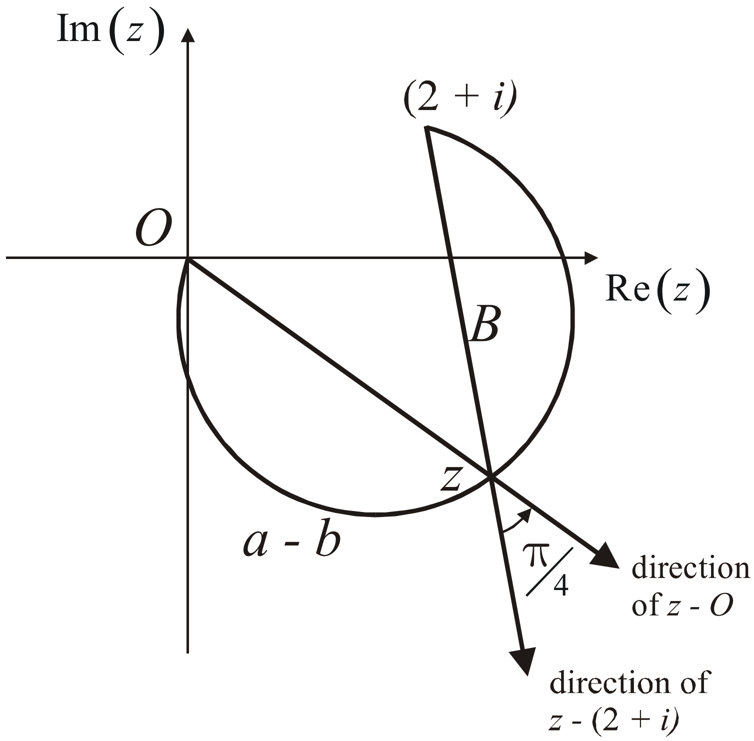
So $\arg\left(\frac{z-o}{z-(2+i)}\right)=\frac{\pi}{4}$ means that the angle between the line joining z to the origin and the line joining z to $(z+i)$ must be $\pi/4$.

In fact, two arcs with the origin and $2+i$ as end- points satisfy this condition.

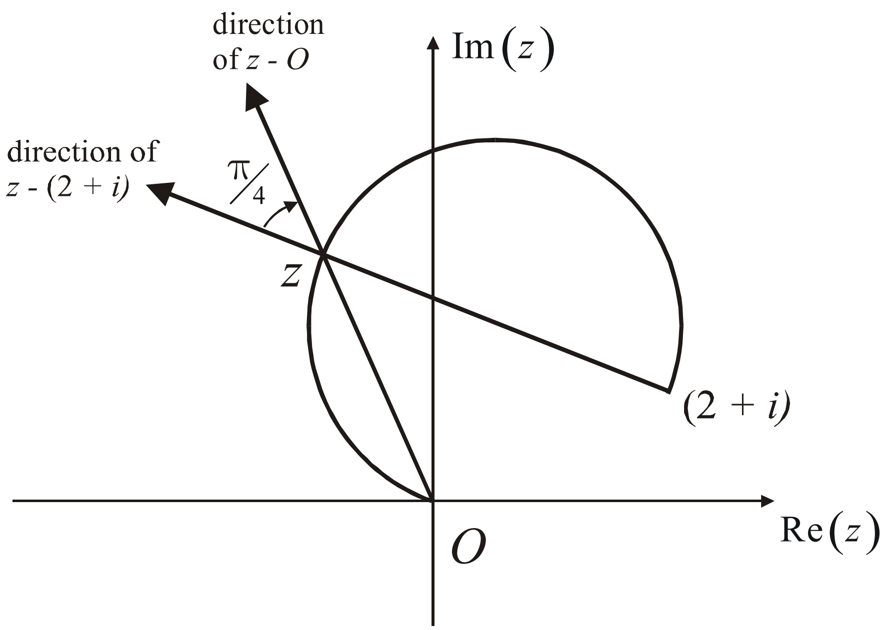


However, in $\arg\left(\frac{z-o}{z-(2+i)}\right)=-\frac{\pi}{4}$ the vector represented by $z-o$ must lead the vector represented by $z-(2+i.)$.

This is satisfied by the lower of the two arcs

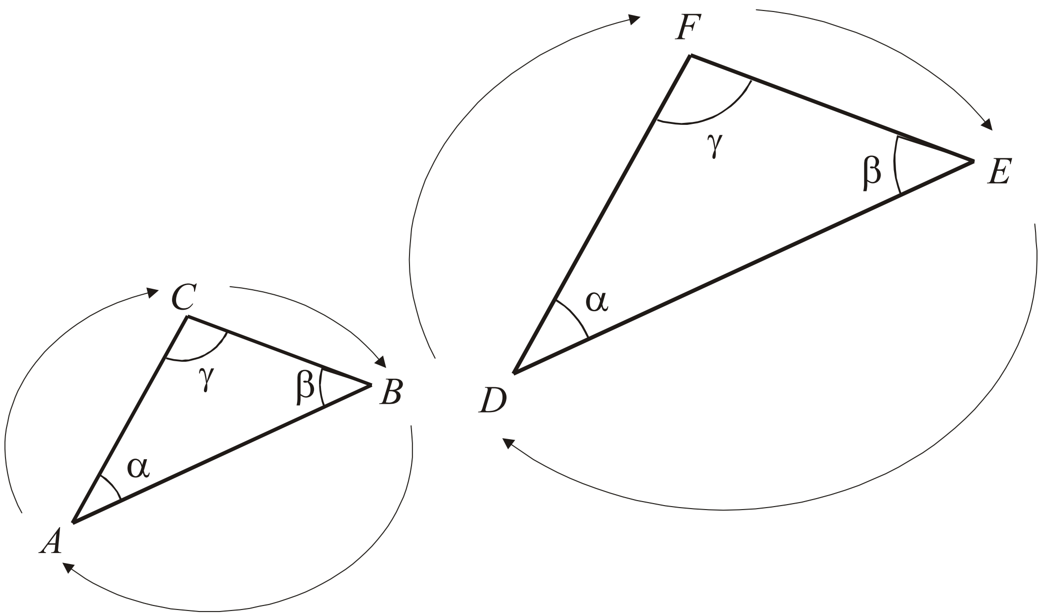


For $\arg\left(\dfrac{z-0}{z-(2+i)}\right)=-\dfrac{\pi}{4}$ this means the vector represented by $z-0$ must trail the vector represented by $z-(2+i)$. This occurs on the other arc,

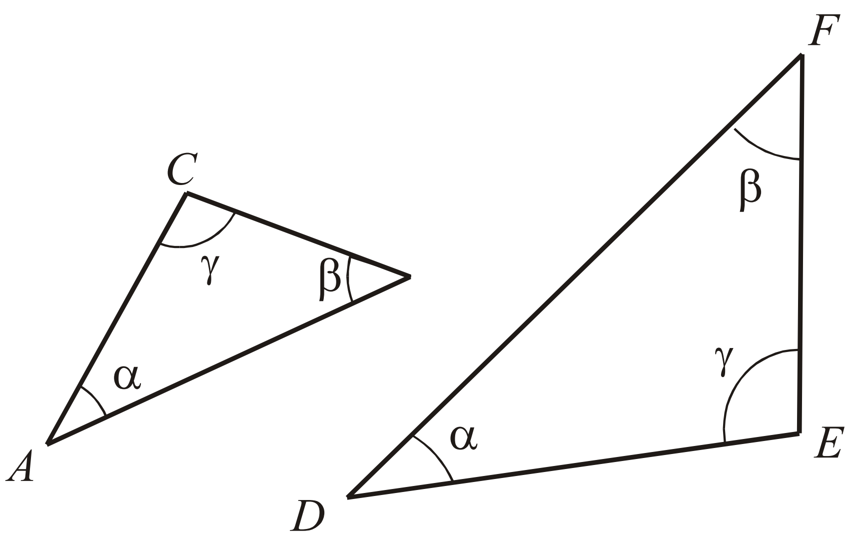


<u>Directly similar triangles</u>

Two triangles are said to be directly similar if (1) they are similar (2) if the points rotate in the same direction.



These two triangles are directly similar because the order in which we take the angles is the same when we proceed in a clockwise (or anti-clockwise) sense.



These two triangles are similar but not directly similar- the order in which we take the angles when proceeding clockwise around both triangles is to the same. In this case they are said to be oppositely similar. Two directly similar triangles can be made to overlap by means of translations, scalings and rotations, but two oppositely similar triangles require a reflection as well.

When points on a triangle are represented by complex numbers there is a criterion for determining when two triangles are oppositely similar; this is

If the ΔABC is represented in the Argand plane by the complex numbers a, b, c and ΔDEF is also represented in the Argand plane by d, e, f, then ΔABC is directly similar to ΔDEF if, and only if

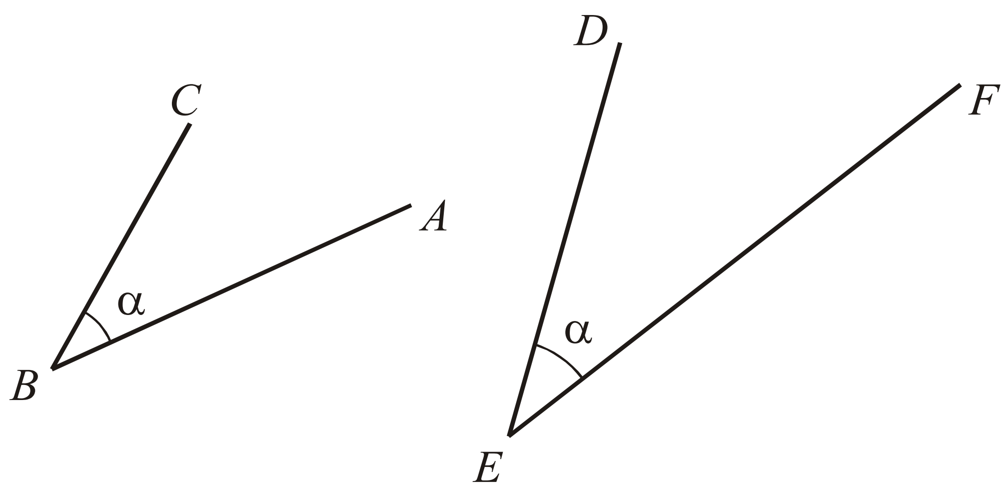
$$\frac{a-b}{c-b} = \frac{d-e}{f-e}$$

Note, the condition $\frac{a-b}{c-b} = \frac{d-e}{f-e}$

implies both $\arg\left(\frac{a-b}{c-b}\right) = \arg\left(\frac{d-e}{f-e}\right)$

and $\left|\frac{a-b}{c-b}\right| = \left|\frac{d-e}{f-e}\right|$

The first property states that the angle between AB and CB must be the same as the angle between DE and FE.



However, this does not fix direct similarity since if AB is shorter than CB it may be that DE is longer than EF.

But if,

$$\left|\frac{a-b}{c-b}\right| = \left|\frac{d-e}{f-e}\right|$$

which means

$$\frac{|a-b|}{|c-b|} = \frac{|d-e|}{|f-e|}$$

then $\frac{AB}{CB} = \frac{DE}{FE}$

and the sides are in the same ratio, so they must be directly similar.

For opposite similarity the direction in which the vertices of one triangle are taken must be reversed. Hence we must invert one of the expressions,

$\frac{a-b}{c-b}$ or $\frac{d-e}{f-e}$

If triangles ABC, DEF are oppositely similar. This gives the criterion

$$\frac{a-b}{c-b} = \left(\frac{d-e}{f-e}\right)*$$

where $*$ stands for the formation of the complex conjugate.