The gradient of a scalar field

Scalar fields

Consider a two- dimensional scalar field $f = f(x, y)$. We will define a vector field called the gradient of the scalar field f by,

$$\text{grad} f = \frac{\partial f}{\partial x}\underline{\mathbf{i}} + \frac{\partial f}{\partial y}\underline{\mathbf{j}} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Example (1)

If $\phi = \ln(x + 3y)$ find grad ϕ. Evaluate grad ϕ at $(1,1)$

Solution

$$\text{grad}\phi = \frac{\partial}{\partial x}\ln(x+3y)\underline{\mathbf{i}} + \frac{\partial}{\partial y}\ln(x+3y)\underline{\mathbf{j}}$$

$$= \left(\frac{1}{x+3y}\right)\underline{\mathbf{i}} + \left(\frac{3}{x+3y}\right)\underline{\mathbf{j}}$$

$$\text{grad}\phi\Big|_{(1,1)} = \frac{1}{4}\underline{\mathbf{i}} + \frac{3}{4}\underline{\mathbf{j}}$$

In three dimensions a scalar field is $f = f(x, y, z)$.

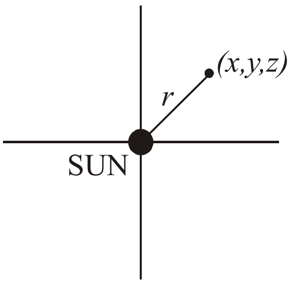
Example (2)

A three dimensional scalar field

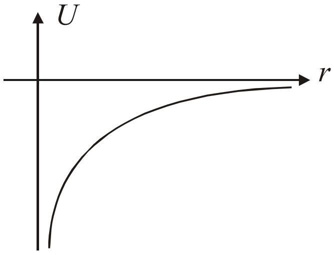
The gravitational potential at a point (x, y, z) of a gravitational field is given by

$$U(x, y, z) = \frac{c}{\sqrt{x^2 + y^2 + z^2}}$$

where c is a constant. For example, the gravitational field surrounding the sun



The gravitational potential is inversely proportional to the distance of the point from the centre of the sun.



Find grad U and show that this points in the direction of the centre of the gravitational field.

<u>Solution</u>

$$U = \frac{c}{\sqrt{x^2+y^2+z^2}} = c\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

$$\frac{\partial U}{\partial x} = -\frac{c}{2}(x^2+y^2+z^2)^{-3/2} \times 2x = c\frac{-x}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial U}{\partial y} = c\frac{-y}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial U}{\partial z} = c\frac{-z}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\therefore \text{grad}U = c\left(\frac{-x}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{-y}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{-z}{\left(x^2+y^2+z^2\right)^{3/2}}\right)$$

$$= -\frac{c}{\left(x^2+y^2+z^2\right)^{3/2}}(x,y,z)$$

So grad U points in the direction $-(x,y,z)$. That is the direction $-\mathbf{i}-\mathbf{j}-\mathbf{k}$

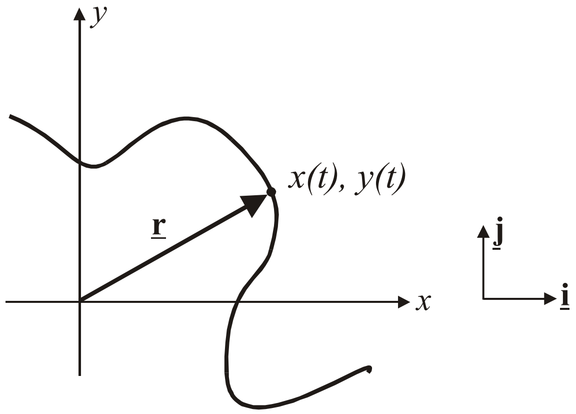
Hence it points towards the centre of the gravitational field.

The direction of grad *f*

We will now show that the direction of grad *f* at a point is perpendicular to contour curve passing through that point – that is, it points in the direction of the normal to that contour.

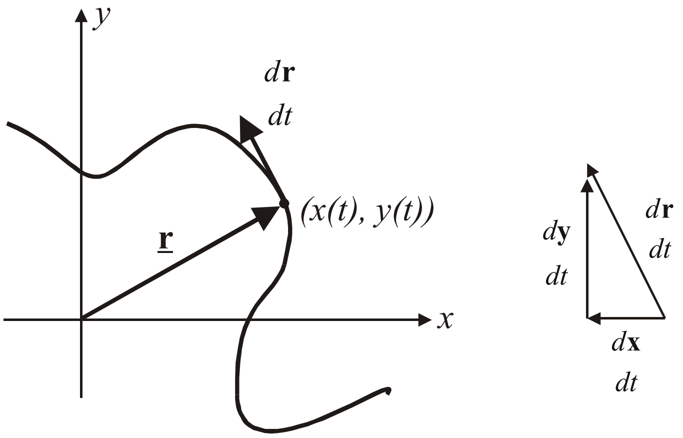
<u>Proof</u>

We have $f(x,y) = k$ as the equation of a contour curve.



Let $\underline{\mathbf{r}} = \left(x(t), y(t)\right)$ be a parameterization of this contour curve.

Then a vector tangent to this contour curve will be $\frac{d\underline{\mathbf{r}}}{dt} = \frac{dx}{dt}\underline{\mathbf{i}} + \frac{dy}{dt}\underline{\mathbf{j}}$.



Along this curve $f(x,y) = f\left(x(t), y(t)\right)$ where k is a constant.

Hence, differentiating with respect to t,

$$\frac{df}{dt} = 0$$

However, since f is a function of x and y and these are regarded as functions of t, we can apply the chain rule to differentiate f.

$$\begin{aligned}\frac{df}{dt} &= \frac{d}{dt}\left(f(x), f(y)\right) \\ &= \frac{df\left(x(t)\right)}{dt}\underline{\mathbf{i}} + \frac{df\left(y(t)\right)}{dt}\underline{\mathbf{j}} \\ &= \left(\frac{df}{dx}\cdot\frac{dx}{dt}\right)\underline{\mathbf{i}} + \left(\frac{df}{dy}\cdot\frac{dy}{dt}\right)\underline{\mathbf{j}}\end{aligned}$$

But here $\frac{df}{dx} = \frac{\partial f}{\partial x}$ is the partial derivative of *f* with respect to x, and likewise $\frac{df}{dy} = \frac{\partial f}{\partial y}$ is the partial derivative of *f* with respect to *y*.

So, $\frac{df}{dt} = \left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} \right) \underline{\mathbf{i}} + \left(\frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \right) \underline{\mathbf{j}}$

Since $\frac{\partial f}{\partial t} = 0$, this means $\left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} \right) \underline{\mathbf{i}} + \left(\frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \right) \underline{\mathbf{j}}$

Now the expression $\left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} \right) \underline{\mathbf{i}} + \left(\frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \right) \underline{\mathbf{j}}$ is the scalar (dot) product of the two vectors $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right), \left(\frac{dx}{dt}, \frac{dy}{dt} \right)$.

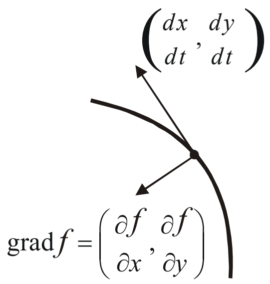
That is $\left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} \right) \underline{\mathbf{i}} + \left(\frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \right) \underline{\mathbf{j}} = \left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt}, \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt} \right)$

Hence the dot product of these two vectors is zero.

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = 0$$

Hence the vector $\left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right)$ is perpendicular to the vector $\left(\frac{dx}{dt}, \frac{dy}{dt} \right)$

Since the vector $\left(\frac{dx}{dt}, \frac{dy}{dt} \right)$ is tangent to the contour curve, the vector $\text{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ is normal to it.



Thus, along a contour curve we have

$$\text{grad } f \bullet \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = 0$$

The surface *f(x, y, z)* = *c*

In three-dimensional space a scalar field is represented by the function $f(x,y,z)$.

A contour or level surface is given by

$f(x,y,z)=c$

Its gradient is

$$\text{grad}f = \frac{\partial f}{\partial x}\underline{\mathbf{i}} + \frac{\partial f}{dy}\underline{\mathbf{j}} + \frac{\partial f}{\partial z}\underline{\mathbf{k}}$$

By adaptation of the above proof to include the extra dimension, we can show that on a contour surface

$$\text{grad } f \bullet \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = 0$$

So grad *f* is normal to the surface.

At a point *P* given by vector **p** the normal line through *P* is given by

$\mathbf{r} = \mathbf{p} + t \text{ grad } f$

Since the tangent plane is perpendicular to this line, the equation of the tangent plane is

$\mathbf{r} = (\mathbf{r} - \mathbf{p}) \bullet \text{ grad } f$

The Vector Operator Nabla

Instead of grad ϕ we can use the expression $\nabla\phi$. The symbol ∇ is pronounced 'del' or 'nabla'. It stands for

$$\nabla\phi = \mathbf{i}\ \frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

And if signifies the operation of finding the first - order partial derivatives of ϕ and the formation of the vector field

∇ is not itself a vector, but by applying ∇ to a scalar field ϕ a vector field is defined; hence it is called a differential vector operator.

Example

Given that $g(x,y,z) = 4x^2 + 3xy + 5y^2 + z^2 + 6z + 1$.

(i) Find $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial z}$.

A surface *S* has equation $4x^2 + 3xy + 5y^2 + z^2 + 6z + 1 = 0$.

(ii) Find the equation of the normal line to *S* at the point $(1,0,\text{-}1)$.

(iii) This normal line meets the surface again at the point *Q*. Find the coordinates of *Q*.

(iv) Find the two values for of *k* for which $8x + 3y + 4z = k$ is a tangent plane to the surface *S*.

Solution

(i) $\frac{\partial g}{\partial x} = 8x + 3y, \quad \frac{\partial g}{\partial y} = 3x + 10y, \quad \frac{\partial g}{\partial z} = 2z + 6$

(ii) We verify that the point $A(1,0,\text{-}1)$ is on the surface. In fact, we have

$$g(1,0,\text{-}1) = 4\times 1^2 + 3\times 1\times 0 + 5\times 0^2 + (\text{-}1)^2 + 6(\text{-}1) + 1 = 4 + 1 - 6 + 1 = 0$$

At A we have $\frac{\partial g}{\partial x} = 8, \quad \frac{\partial g}{\partial y} = 3, \quad \frac{\partial g}{\partial z} = 4$

Therefore, $\text{grad}\, g = \begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix}$

The normal line at $A(1,0,\text{-}1)$ is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 8 \\ 3 \\ 4 \end{pmatrix}, \quad t \in \mathbb{R}$.

(iii) From (ii) we have that

$$(n) \quad \begin{cases} x = 1 + 8t \\ y = 3t \\ z = -1 + 4t \end{cases}$$

This line (n) meets again the surface S and this point is obtained from the following system

$$\begin{cases} x = 1 + 8t \\ y = 3t \\ z = \text{-}1 + 4t \\ 4x^2 + 3\cdot x\cdot y + 5y^2 + z^2 + 6z + 1 = 0. \end{cases}$$

Therefore $4\times(8t+1)^2 + 9\times(8t+1)\cdot t + 5\times 9\times t^2 + (4t-1)^2 + 6\times(4t-1) + 1 = 0$

$$256t^2 + 64t + 4 + 72t^2 + 9t + 45t^2 + 16t^2 - 8t + 1 + 24t - 6 + 1 = 0$$

$$389t^2 + 89t = 0 \quad \Rightarrow \quad t_1 = 0 \quad \text{and} \quad t_2 = -\frac{89}{389}$$

Therefore $Q\left(1 - \frac{8\cdot 89}{389}, \frac{\text{-}89\cdot 3}{389}, \text{-}1 - \frac{89}{389}\cdot 4\right)$, i.e. $Q\left(-\frac{323}{389}, -\frac{267}{389}, -\frac{745}{389}\right)$

(iv) We need $\frac{8x+3y}{8} = \frac{3x+10y}{3} = \frac{2z+6}{4} = k_0$

(1) at the point (x,y,z) in which the tangent plane is of form $8X + 3Y + 4Z = k$.

From (1) we obtain $\begin{cases} 8x + 3y = 8k_0 \\ 3x + 10y = 3k_0 \\ 27 + 6 = 4k_0 \end{cases}$

Therefore $\begin{cases} -24x - 9y = -24k_0 \\ 24x + 80y = 24k_0 \end{cases} \oplus$

$y = 0$, $x = k_0$ and $z = 2k_0 - 3$, i.e. $(x, y, z) = (k_0, 0, 2k_0 - 3)$

This point satisfies the equation $4x^2 + 3xy + 5y^2 + z^2 + 6z + 1 = 0$, i.e.

$4k_0^2 + (2k_0 - 3)^3 + 6 \cdot (2k_0 - 3) + 1 = 0$, i.e.

$8k_0^2 - 12k_0 + 9 + 12k_0 - 18 + 1 = 0$

$8k_0^2 - 8 = 0 \quad \Rightarrow \quad k_0 = \pm 1.$

I. $k_0 = -1$, $(x, y, z) = (-1, 0, -5)$

Therefore $k = 8 \cdot (-1) + 3 \cdot 0 + 4 \cdot (-5) = -28$

So, $8x + 3y + 4z = -28$

II. $k_0 = 1$, $(x, y, z) = (1, 0, -1)$

Therefore $k = 8 \cdot 1 + 3 \cdot 0 + 4 \cdot (-1) = 4$

So, $8x + 3y + 4z = 4$