Introduction to graph theory

Graphs

A graph is a structure in which points, called vertices, are joined by lines, called edges. A vertex is also called a *node.* From this idea arises a series of terms that are used to define graphs and their properties.

Vertex
A vertex is a point, or node.

Edge
An edge is a line joining two vertices. An edge is also called an *arc.*

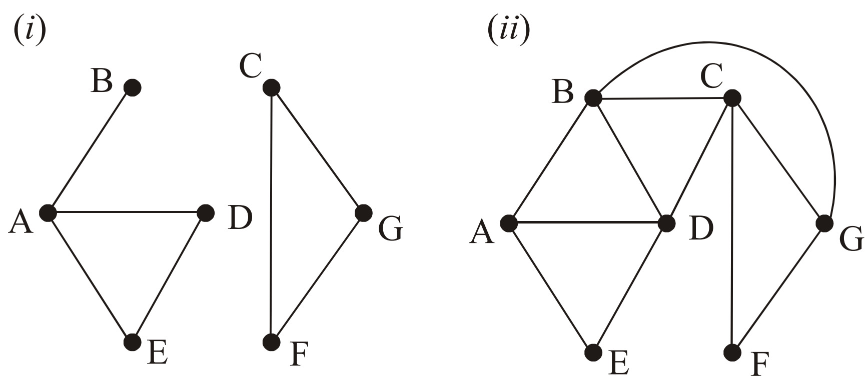
Graph
A graph is a line drawing connecting vertices to one another by means of edges.

Path
A path is a sequence of vertices such that each pair of successive vertices is joined by an edge.

Connected graph
Two points on a graph are said to be connected if a path can be found from one point to the other.

Example

Which of these two graphs is connected?



Solution

The graph (*i*) is not connected. The graph (*ii*) is connected.

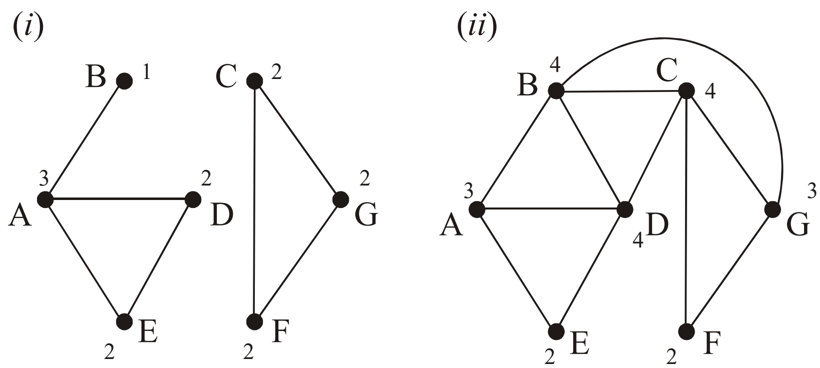
Order of a vertex
The number of edges meeting at a vertex is called the order of the vertex.

Example

Find the order of each vertex for each of the two graphs above.

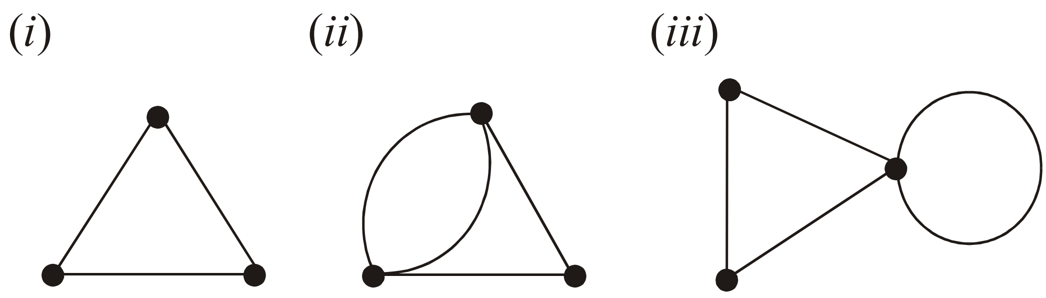
Solution

The orders are marked on the graphs



Simple graph
A simple graph contains no loops or multiple connections.

For example

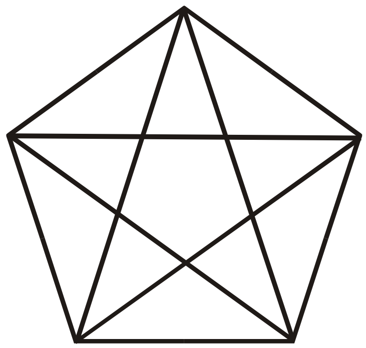


Simple Not simple Not simple

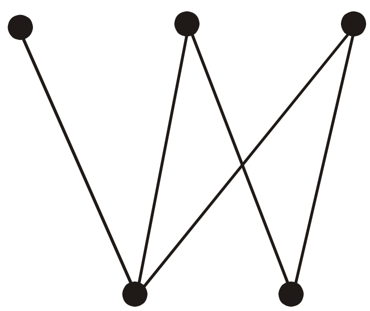
Complete graph
A complete graph is simple and every pair of vertices is joined by an edge.

The complete graph of n vertices is called K_n.

For example, K_5



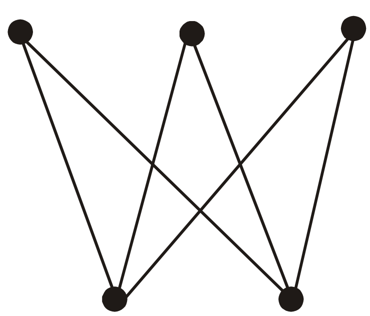
Bipartite graph
A bipartite graph is a graph where the nodes (vertices) are divided into two subsets, and there are no connections between the nodes within the separate subsets. For example, this is a bipartite graph



Complete bipartite graph
A complete bipartite graph will be a graph that is both bipartite and complete! It has a special symbol. For example, the symbol

$K_{3,2}$

denotes the following complete bipartite graph



Subgraph
A subgraph is a graph contained within a graph.

For example, the first bipartite graph given above is a subgraph of the second (complete) bipartite graph.

Maps and Euler's Theorem

Isomorphism

The term isomorphic is used to indicate when two graphs are "the same". When two graphs are the same, we say there is an *isomorphism* between them. However, what being the same means needs further clarification. Essentially, two graphs will be the same if their edges can be "pulled" around so that they come to coincide. What this means in practice is clearer with examples – as explained in the next section on planar graphs.

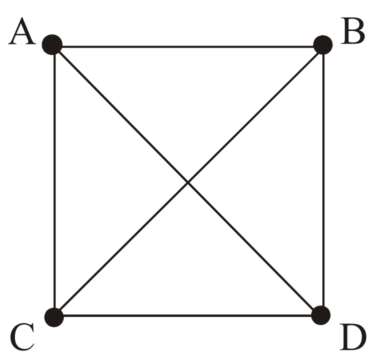
Planar graph

A plane, or planar graph is one which can be drawn in such a way that the edges do not cross. It is isomorphic to a graph where the edges do not cross.

This definition indicates that sometimes a graph which is planar does not appear to be planar, owing to the way it has been drawn. To prove that it is planar, you have to show that it is isomorphic to a planar graph.

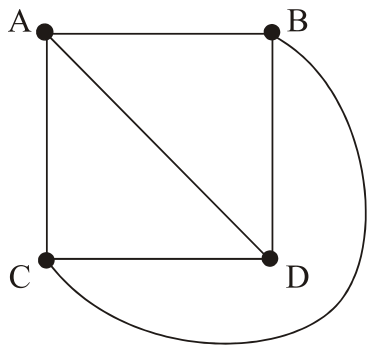
Example

Show that the following graph is planar



Solution

If we "pull out" the line CB, we can transform this to a graph which is planar.

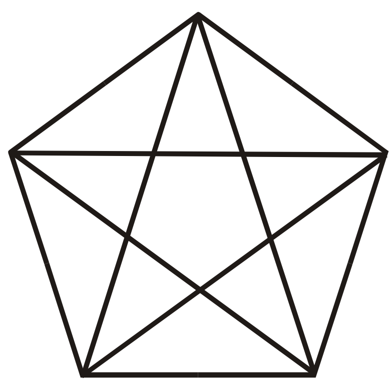


Hence the graph is isomorphic to a planar graph, and is consequently planar.

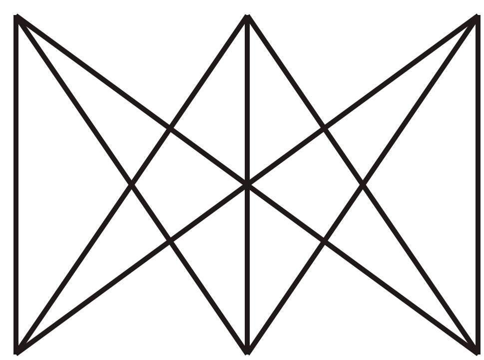
<u>Map</u>
A map is a representation of a finite, planar graph. A map is another name for a finite, planar graph is a map.

Kuratowski's Theorem

This is the theorem that a graph is nonplanar if and only if it contains a subgraph that is isomorphic to either $K_{3,3}$ or K_5.



K_5



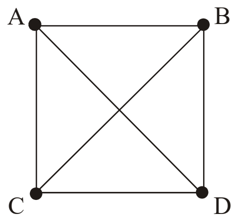
$K_{3,3}$

Edges and regions

<u>Regions</u>
Maps divide planar graph into regions. Regions are also called faces. The borders of regions are edges.

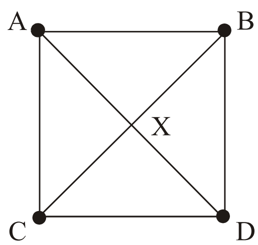
Example

How many regions does the following graph have?

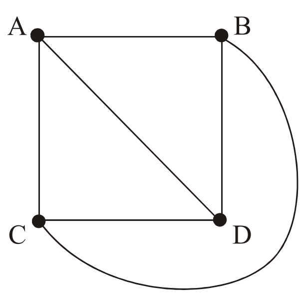


Solution

The answer would appear to be four, but this would be an error arising from the particular way the graph has been drawn. The point where the edges AD and BC cross in this diagram is not a vertex of the graph. If we label this point temporary by X



Then the figure AXC, for example, does not define a region, since AX and CX are *not* edges of this graph. In order to count accurately the number of regions, we must first represent it properly as a map – that is, as a planar graph. We saw such a representation in the preceding example



Now we can see that there are 3 regions contained within the figure, but also there is a region contained outside the figure. This will become clearer as we discuss this example further in what is to come. However, there are four regions in all.

Degree of a region

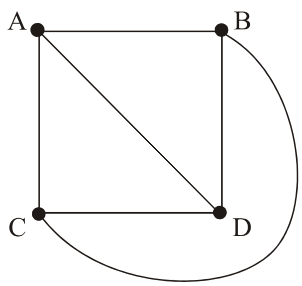
Each region in a map is bounded by a cycle of edges. The number of edges in the boundary of a region is the degree of the region.

Note that a map is drawn on some surface. The region beyond the outermost edges of the graph, but lying on the surface, is also counted as a region. It has no boundary on the "outside", but is bounded by those edges of the map that meet it. When counting the regions of a map, we must include this region as well.

Another point is that an edge that connects to a vertex of degree 1 adds two boundaries to the region that borders it.

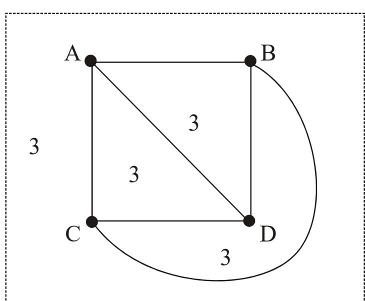
Example

Find the degree of each region of the following map



Solution

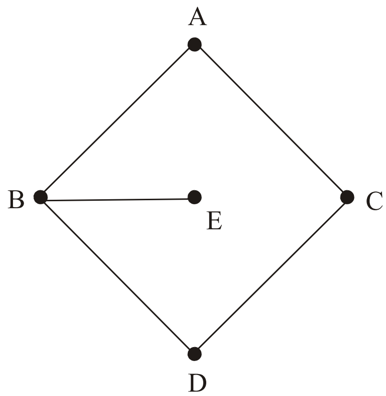
Each region is bounded by three edges. The degree of each region is 3.



The outermost region also has degree three. We picture it by drawing in a rectangle around the entire graph. Since this rectangle does not represent a bounded surface, we show it by a dashed line. The figure is not bounded on the "outside", but on the "inside" it is bounded by the edges AC, AB and CB, and has degree 3.

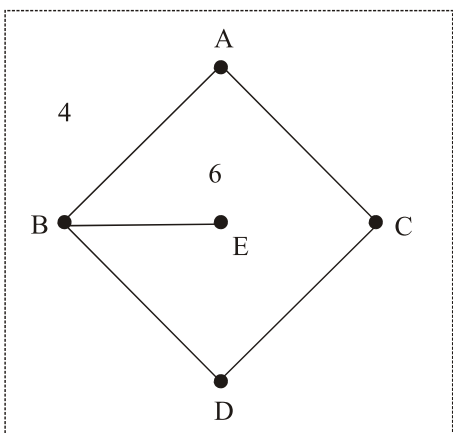
Example

Find the degree of each region of the following map.



The edge BE counts as two boundaries from the point of view of the number of edges in the boundary of the region on the inside of the rectangle. Working clockwise around this region we have the edges AC, CD, DB, BE, EB, AB. In other words, we have to move out along the edge BE to reach the vertex E, and then back along EB to reach B. So the edge counts twice in this respect.

The degrees of the regions are consequently

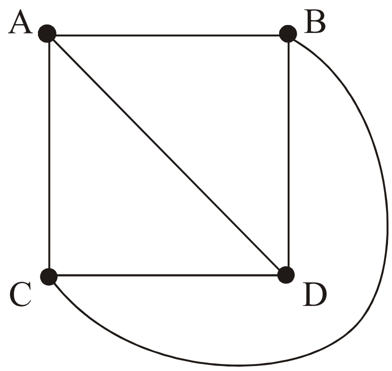


Theorem on edges and regions

The sum of the degrees of the regions of a map is equal to twice the number of edges.

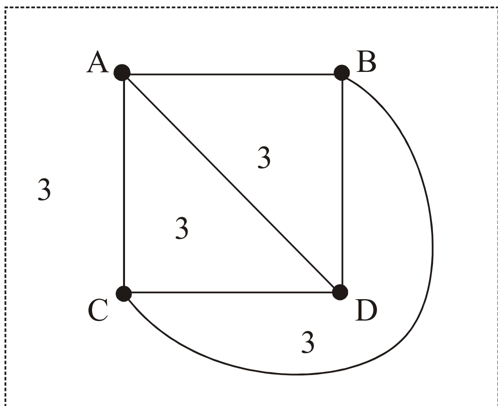
Example

Verify the edges/regions theorem for the map



Solution

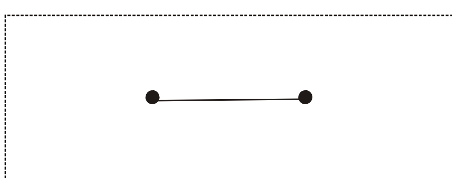
We just saw that the regions of this map each have degree 3.



The sum of the regions is, therefore, 12. There 6 edges. Hence, the theorem certainly applies to this map.

Proof of the edges/regions theorem

Starting with a planar graph comprising a single edge.



There is just one edge. The region surrounding this edge has degree 2. Hence, the theorem holds for this figure.

Suppose the theorem holds for a figure of k edges. Now imagine adding one more edge. If the edge connects two vertices it will divide a region in two – into two new regions. In so doing it will add two boundaries to the total number of boundaries – one will be added to one of the new regions and one to the other. Hence the total degree of the regions of the figure will increase by two.

It the edge does not connect two vertices, so that one vertex already in the graph is connected to a new vertex added at the same time the edge is added, then that edge will add two boundaries to the region that it "butts" into.

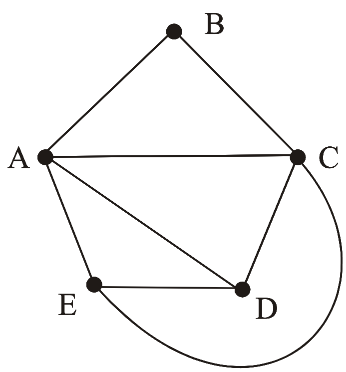
In other words, the theorem will hold for all planar graphs (or maps).

<u>Euler's formula</u>

For a connected plane graph with V vertices, E edges and R regions:
$V - E + R = 2$

<u>Example</u>

Verify that Euler's formula holds for the planar graph

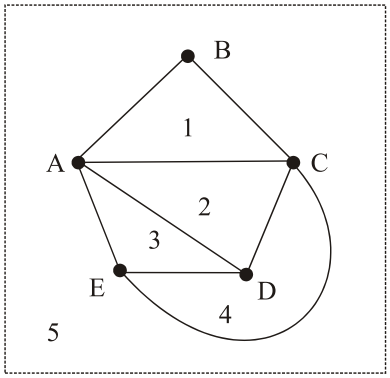


Solution

There are 5 vertices, 8 edges and 5 regions, hence

$V = 5,\ E = 8,\ R = 5$
Hence,
$V - E + R = 5 - 8 + 5 = 2$
as expected.

The regions of this figure are counted as follows

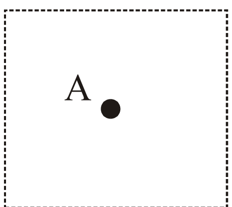


The main point is that the region contained with the rectangle and bounded by the edges AB, BC, CE and AE must also be counted. With this region counted in, Euler's formula holds.

Proof of Euler's theorem

The simplest planar graph comprises a single vertex, A. For this figure,

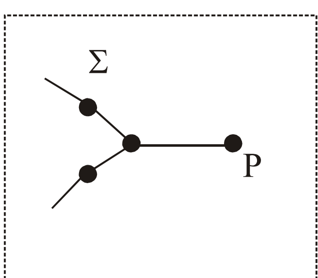
$V = 1,\ E = 0,\ R = 1$



Now suppose that Euler's formula holds for a planar graph, Σ. A new graph can be constructed by adding an edge in one of two ways.

(1)

Add a new vertex and connect that vertex by the new edge to the existing planar graph.



This will increase the edges by 1, the vertices by 1 and the regions by 0. Hence, the new value of

$V - E + R$

will not be altered. Hence, it will remain the case that $V - E + R = 2$.

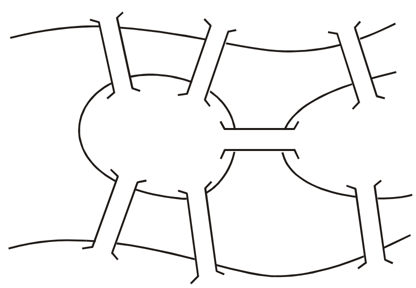
(2)

Add an new edge and connect two existing vertices. This increases the edges by 1, the vertices by 0, and the regions by 1. Hence, as before $V - E + R = 2$

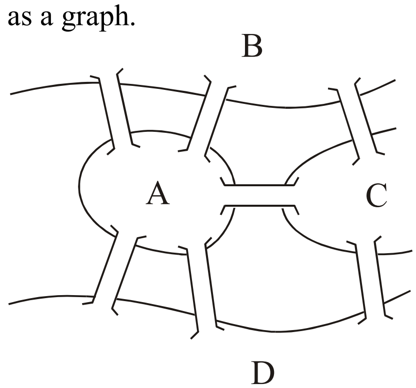
The Bridges of Königsburg

The people of Königsburg were interested in whether it was possible to walk through the town of Königsburg crossing all seven bridges. They sent a map to the famous Swiss mathematician, Euler, who invented graph theory as a result.

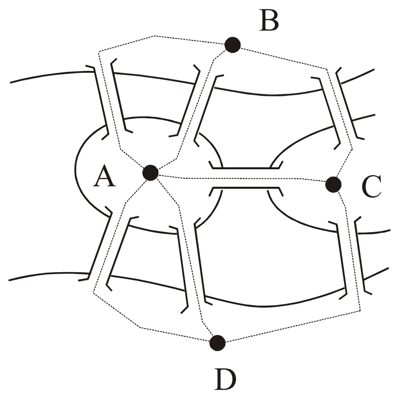
The seven bridges connect two islands and two river banks in the following way.



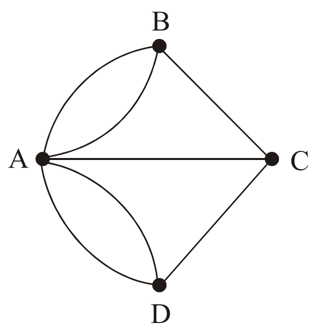
By labelling the solid areas and treating them as points, Euler was able to redraw this as a graph.



This puts in the labels. We now turn the banks and regions into points, and draw in the edges.



This gives the graph



The graph of the bridges of Königsburg problem is also an example of a multigraph.

<u>Multigraph</u>
A graph is a multigraph if at least two nodes have more than one edge joining them.

As we can see, in this graph there are two ways to get from A to B and two ways to get from A to D.

Note, we will use the term <u>graph</u> to encompass all graphs, including multigraphs. Thus all multigraphs are graphs, but not all graphs are multigraphs.

We need to add some yet further definitions.

Recall that a path is sequence of edges connecting vertices.

<u>Simple path</u>
This is a path in which all the vertices are distinct (different) – no vertex is visited twice.

<u>Trail</u>
A trail is a path in which all the edges are distinct – no edge is used more than once.

<u>Closed</u>

A path is closed if the starting vertex is equal to the finishing vertex.

Cycle

A closed path is a cycle if all the vertices are distinct except the starting and finishing vertex, which are identical. It is also called a circuit.

Closed trail (path)

Closed means the trail through the graph finishes where it starts.

Traversable trail (path)

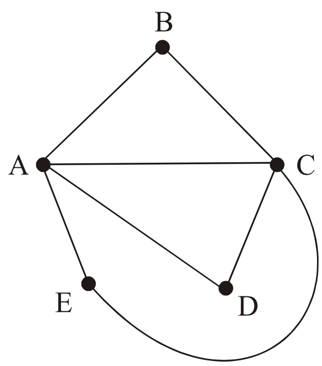
A traversable graph is one that can be drawn that the trail can be drawn without lifting the pen from the paper, and without going over any of the edges.

The question the inhabitants posed to Euler was effectively, was this graph “closed” and “traversable”.

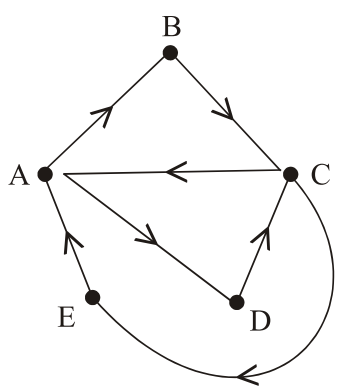
Eulerian trail

We call a graph Eulerian if it is closed and traversable.

An example of a closed, traversable trail, that is, and Eulerian trail, is



To show that it is closed and traversable we construct a suitable trail



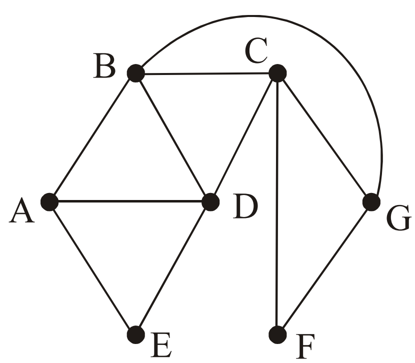
Euler was able to demonstrate that there could not be a solution to the bridges of Königsburg problem – that is, it is not possible to find a closed, traversable trail (an Eulerian trail) for the bridges of Königsburg graph.

A vertex is even if its degree is even. This means that there is an even number of edges joining that vertex to other vertices. Similarly, a vertex is odd if its degree is odd.

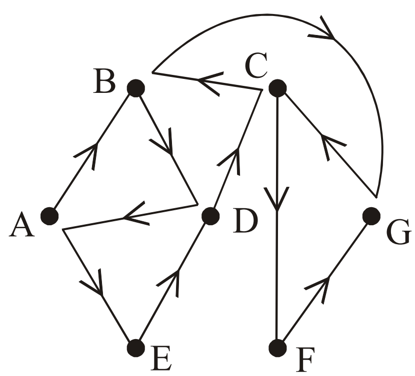
If a graph is traversable, then for any vertex other than the starting or ending vertex the vertex must be even.

If that vertex is not the starting or ending vertex, every time that vertex is visited a new during a trail, then there must be both an entry and an exit edge. A trail is only traversable if the edges in the graph are only used once, so this means that the vertex must, in this case, be even.

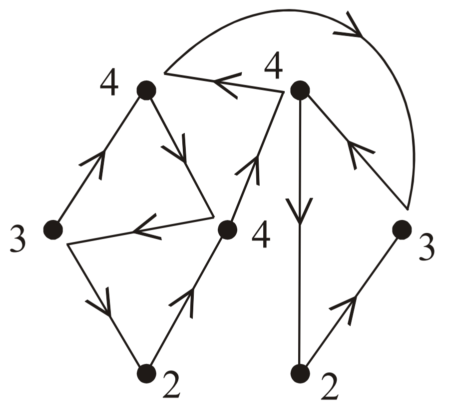
If a vertex is odd it must be the first or last vertex. The following is a traversable trail, where the beginning and ending vertices are odd.



A traversable trail through this graph is



The order of each vertex is



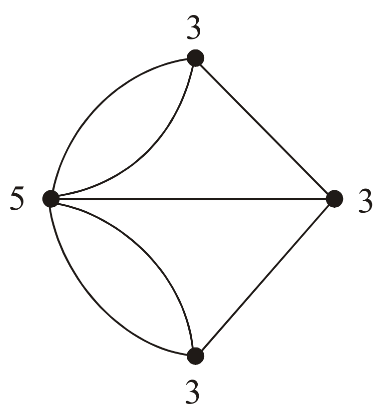
If a graph is closed and traversable, then the starting vertex is the same as the end vertex. Consequently, a closed, traversable graph can have at most one odd vertex.

But actually, we can now prove that the starting vertex must have even degree. The point here is that since the graph is now closed and traversable, there is nothing unique about the starting (and also ending) vertex. Any vertex will do for a starting or ending point. So every vertex lies on the trail of some other vertex. But we just showed that vertices that are within a traversable trail must be even. Since this is true of all vertices, all the vertices must be even.

In other words, a graph will be closed and traversable if, and only if, every one of its vertices is even.

This means that if only one of the vertices is odd, then the graph will not be closed and traversable. It also means that if all the vertices are even, then the graph must be closed and traversable.

In the Königsberg bridge graph all the vertices are odd. Their orders are



Hence, there cannot be a solution to the Königsberg bridge problem.

Recall that we call a graph Eulerian if it is closed and traversable, so the main result of this sub-section could be rephrased:

A finite connected graph is Eulerian if, and only if, each vertex has even degree.

We have also shown that any finite graph with exactly two odd vertices is traversable but not closed. The trail that traverses such a graph would have to begin at one of the odd vertices and end at the other.

Trace
A trace is a sequence of vertices, and a trace may be named by listing its vertices in order.

Eulerian/Semi-Eulerian
A trace which includes every edge is called an *eulerian tail* if it is closed – that is finishes where it started; and a *semi-eulerian tail* if it is not closed.

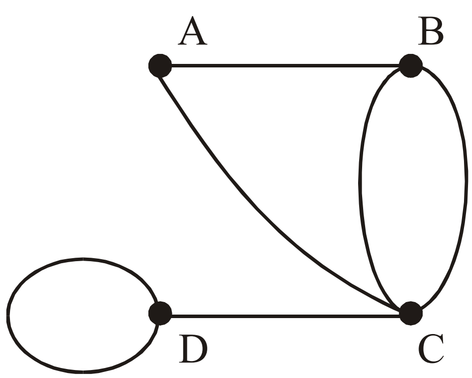
So semi-eulerian is just another term for a traversable graph that is not closed.

Hamiltonian circuit
A Hamiltonian circuit is a closed circuit (cycle) that visits every vertex once.

Hamiltonian graph
If a graph is such that it contains a Hamiltonian circuit then it is a Hamiltonian graph.

Incidence tables and isomorphic graphs

Incidence tables
Vertex-vertex incidence table is best introduced by example



	A	B	C	D
A	0	1	1	0
B	1	0	2	0
C	1	2	0	1
D	0	0	1	1

The incidence table on the left records the number of connections (edges, arcs) between two vertices (nodes). Thus, there is one edge connecting A to B so the entry in the row for A and the column for B is a 1. This also means that there is a 1 in the entry for the row for B and column for A. In other words the incidence table is symmetric about the main diagonal.

A is not connected to itself, so the entry in the row for A and column for A is a 0.

There are two edges connecting B and C, so the entry in the B*th* row and C*th* column is a 2.

Thus this is a multigraph, since there are two edges connecting B to C. Furthermore, it is a graph with a loop.

Loop
A loop is an edge whose starting and finishing vertex are the same.

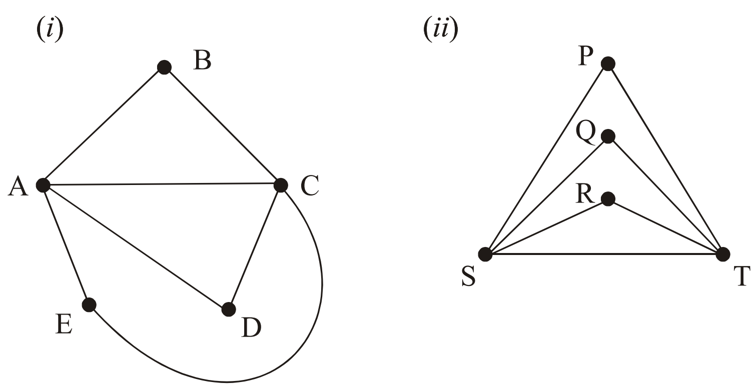
Thus the edge connecting D to itself is a loop. The presence of a loop in an incidence table is shown by the entry of a 2 in the diagonal of the table.

Simple graph
A simple graph is a graph without multiple edges or loops. The term multigraph encompasses all graphs with either multiple edges or loops or both.

Isomorphic graphs
When two graphs show the same connectedness they are said to be isomorphic and are regarded as being the same in graph theory. To demonstrate that two graphs are isomorphic one can show that their tables are the same.

Example

Demonstrate that the following graphs are isomorphic



Solution

Firstly, we write the incidence tables for the two graphs

(*i*)

	A	B	C	D	E
A	0	1	1	1	1
B	1	0	1	0	0
C	1	1	0	1	1
D	1	0	1	0	0
E	1	0	1	0	0

(*ii*)

	P	Q	R	S	T
P	0	0	0	1	1
Q	0	0	0	1	1
R	0	0	0	1	1
S	1	1	1	0	1
T	1	1	1	1	0

Now we have to bring them to look the same. To do this we can swop rows, (or equivalently columns).

If we swap A and C in the first incidence table we obtain

(*i*)*

	E	B	C	D	A
E	0	0	1	0	1
B	0	0	1	1	1
C	1	1	0	1	1
D	0	0	1	0	1
A	1	1	1	1	0

Then swapping C and D gives an incidence table that can be compared directly with that of (*ii*).

(*i*)**

	E	B	D	C	A
E	0	0	0	1	1
B	0	0	0	1	1
D	0	0	0	1	1
C	1	1	1	0	1
A	1	1	1	1	0

(*ii*)

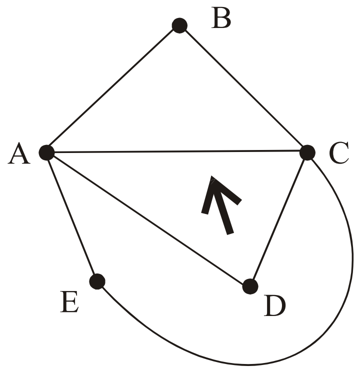
	P	Q	R	S	T
P	0	0	0	1	1
Q	0	0	0	1	1
R	0	0	0	1	1
S	1	1	1	0	1
T	1	1	1	1	0

So we can see they are isomorphic.

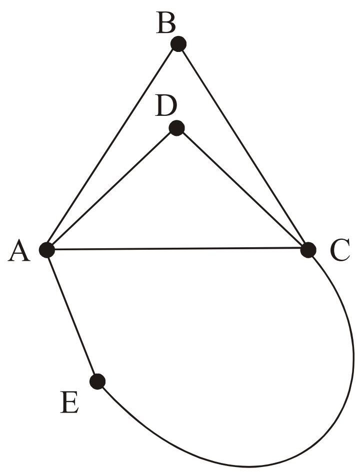
Another way of demonstrating that two graphs are isomorphic is to manipulate their edges and vertices until they look exactly the same.

Example continued

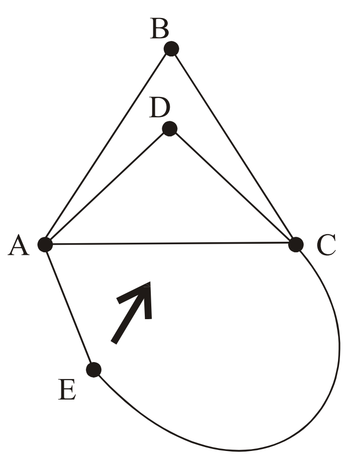
If in this example we pull the vertex D in this direction



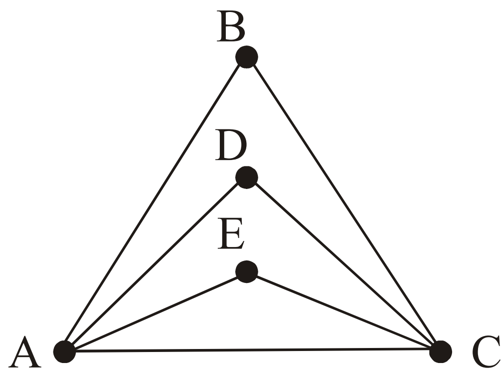
we obtain



Now pulling E in this direction



We obtain



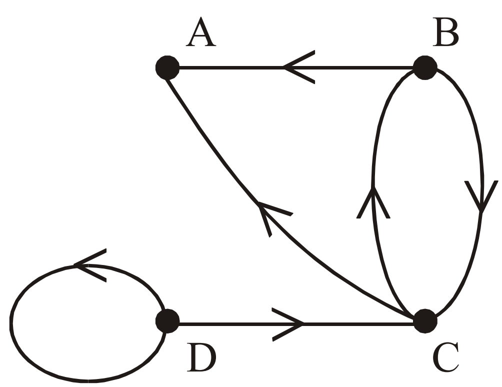
Which is isomorphic to (*ii*).

Definitions of digraphs, networks and trees

Directed edge
Edges in graphs may be directed – that is orientated.

Digraph
A directed graph is called a *digraph*.

Incidence tables for digraphs are illustrated as follows



	A	B	C	D
A	0	0	0	0
B	1	0	1	0
C	1	1	0	0
D	0	0	1	1

An entry of a “1” in row *X* and column *Y* indicates that *X* is connected by a directed edge to *Y*.

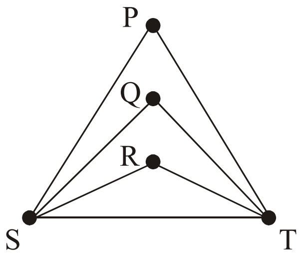
Digraphs are isomorphic when they may be represented by the same table.

Weighted graph - Network
A weighted graph or weighted digraph is one in which a number is associated with each edge. Weighted graphs and digraphs is also called a network.

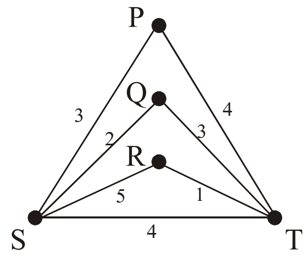
Example

Which of these is a network?

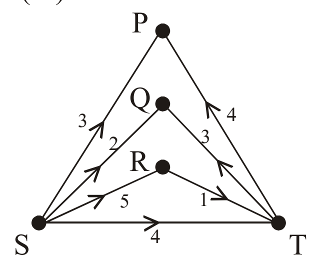
(*i*)



(*ii*)



(*iii*)



Solution

(*i*) is not a network; both (*ii*) and (*iii*) are. (*ii*) is a weighed graph, and (*iii*) is a weighted digraph.

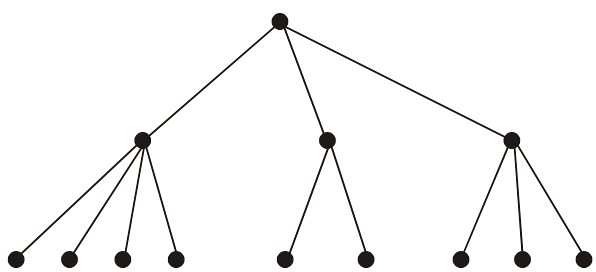
Digraphs are studied in a further unit.

Trees

Trees are connected graphs that have no circuits.

Recall that a circuit is a closed path – it starts and ends on the same vertex.

This is an example of a tree.

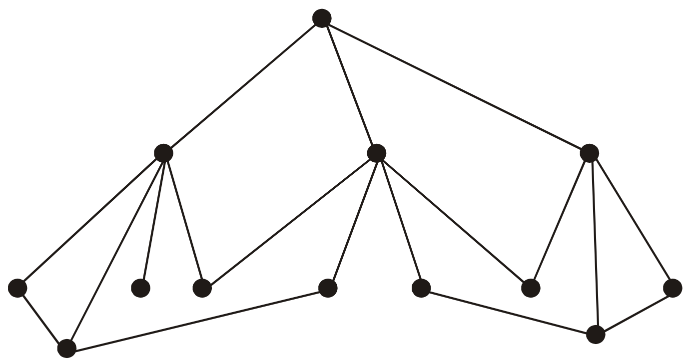


If a connected graph has a circuit, it can be deleted without disconnecting the graph.

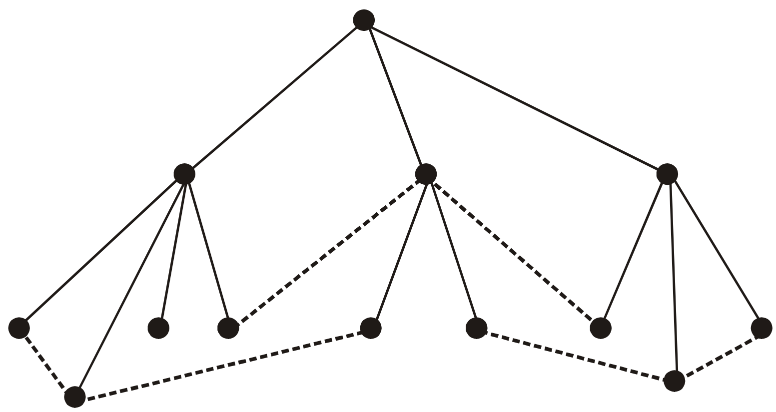
Spanning tree

Deleting edges in such a way that circuits are removed will result in a tree, which connects all the vertices of the original graph. Such a tree is called a spanning tree.

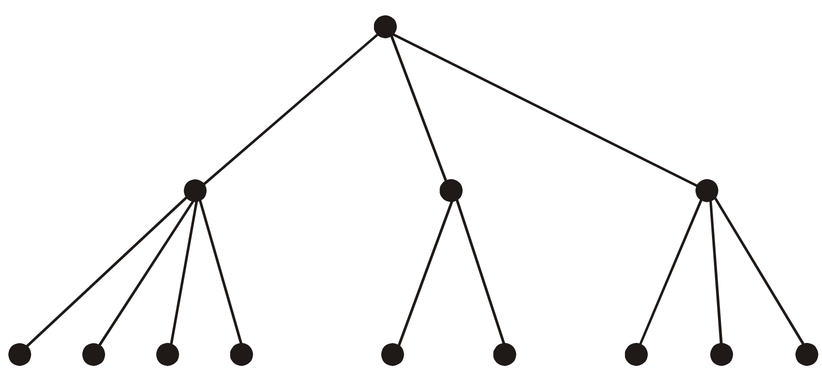
For example, the following graph has circuits in it.



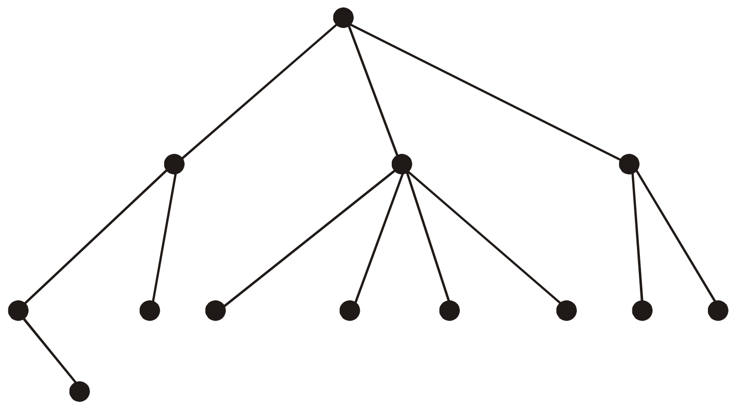
Deleting the following edges will produce a tree.



In this case it is the same tree as the one before.



However, if other edges are deleted, we obtain a different tree, which equally spans the graph.



If the graph is weighted (that is, if it is a network), one of these trees may be of less total weight than the other. In this case, there will be a minimal spanning tree.

The subject of trees and minimal spanning trees is taken up in a subsequent unit.