Graphs of Rational Functions

Rational functions - prerequisites

A rational function takes the form

$$h(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomials. By this stage you should be familiar with (a) the technique of polynomial division and the use of the remainder and factor theorems to find roots of polynomials; (b) the technique of partial fractions; (c) analytic methods of finding stationary points – that is, the use of the differential calculus to identify maxima, minima and points of inflection; (d) sketching graphs of polynomial functions using such analytic information. In this chapter you will learn how to apply your knowledge of rational functions gained from these techniques to sketch graphs of rational functions.

Graphs of rational functions

Graphs of rational functions exhibit the following features:

(1) Turning points and points of inflection – that is, points where the gradient of the tangent to the curve of the function is zero – which are maxima, minima and points of inflection;

(2) Roots – that is, points where the function takes the value 0, and hence crosses the *x*-axis (the axis of the independent variable);

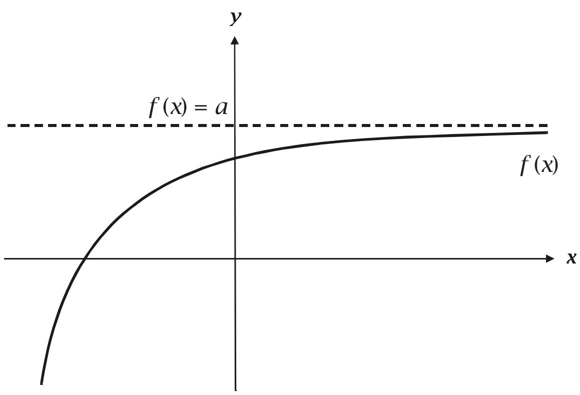
(3) Asymptotes – these are lines to which the graph of the function gets closer and closer as the independent variable, *x*, gets closer and closer to some value. The graph gets closer and closer to these lines without actually reaching them.

Techniques for finding turning points make use of the differential calculus to determine the points where the gradient function

$f'(x) = 0$.

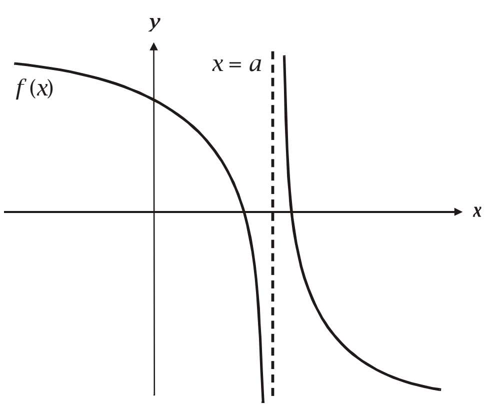
Techniques for finding roots use either direct factorisation or use a numerical method. We are now concerned with asymptotes. Asymptotes can be horizontal, vertical or oblique.

Horizontal asymptote



In this diagram as the function $f(x)$ approaches $+\infty$ it gets closer and closer to a certain value, *a*. We could write this as: $x \to +\infty \quad f(x) \to a$ and another notation for the same idea is $\lim_{x\to\infty} f(x) = a$. Here the expression "lim" is short for limit. We say in the limit, as *x* tends to $+\infty$, $f(x)$ tends to a. In other cases the function might approach an asymptote as *x* tends to $-\infty$, and it might approach it from above or below. We need to be able to recognise these cases from the algebraic expression for the function $f(x)$.

Vertical asymptote



In this example as the function gets closer and closer to a certain value *x* = *a* the function takes ever increasing negative values from one side, and ever increasing positive values from the other.

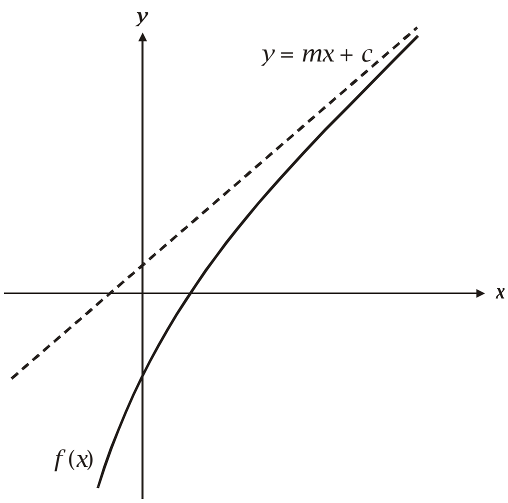
At the point $x = a$ the function takes no value whatsoever and is undefined[1]. We say there is a *singularity* at the point $x = a$. Using the notation of limits we write this as

$$x \to +a \quad f(x) \to -\infty \qquad \lim_{x \to a} f(x) = -\infty$$

$$x \to -a \quad f(x) \to +\infty \qquad \lim_{x \to a} f(x) = +\infty$$

The use of the symbols $+, -$ in $x \to +a$ or $x \to -a$ indicates from which side of the point $x = a$ we are approaching. The point $x = a$ is called a *singularity* of the function. Once again, we need to be able to recognise singularities from the algebraic form of the function.

Oblique asymptote



In this diagram we see an example of a function with an oblique asymptote. That is, as x tends to infinity, the graph of the function gets closer and closer to a line with the equation $y = mx + c$ (where m and c are parameters to be determined). We write this

$$x \to +\infty \quad f(x) \to mx + c \qquad \lim_{x \to +\infty} f(x) = mx + c$$

Sketching graphs of rational functions

In order to sketch the graph of a rational function we must first use the technique of partial fractions to write it as a sum of fractions where each denominator is smaller than the numerator.

[1] An exception is that if the function is defined *piecewise* it may be arbitrarily assigned a value at $x = 0$.[1] However, this does not affect the graph sketching techniques under discussion here, since the arbitrary value would be represented in the graph simply as a point For more information about defining functions piecewise, please read our chapter *Functions and Continuity*

The presence of each type of asymptote can be detected from the form that a rational function takes once it has been decomposed into partial fractions.

Example (1)

Write $\frac{x^3}{x^2-4}$ as a sum of partial fractions.

Solution

Using polynomial division

$$\begin{array}{r}
x \\
x^2-4\overline{)x^3+0x^2+0x+0} \\
\underline{x^3 \qquad -4x} \\
4x
\end{array}$$

or simply by inspection

$x^3 \equiv (x^2-4)x+4x$

Then

$$\frac{x^3}{x^2-4} \equiv x + \frac{4x}{x^2-4}$$

Breaking into partial fractions

$$\frac{4x}{x^2-4} \equiv \frac{4x}{(x-2)(x+2)} \equiv \frac{A}{x-2} + \frac{B}{x+2} \equiv \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

$A(x+2)+B(x-2) \equiv 4x$

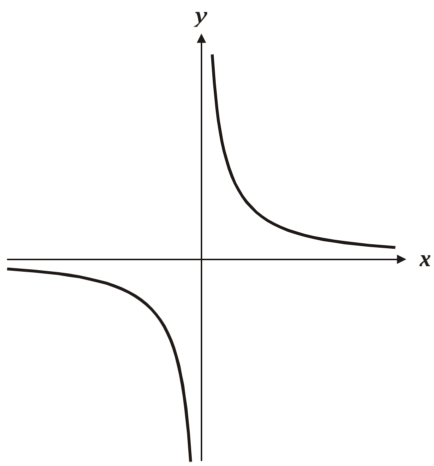
$x=-2 \Rightarrow -4B=-8 \Rightarrow B=2$

$x=2 \Rightarrow 4A=8 \Rightarrow A=2$. Hence

$$\frac{x^3}{x^2-4} \equiv x + \frac{2}{x-2} + \frac{2}{x+2}$$

Horizontal Asymptote

A horizontal asymptote occurs when $f(x)$ tends to a certain constant value as x tends to either $+\infty$ or $-\infty$. An example is the rectangular hyperbola; that is, $y = 0$ is the horizontal asymptote of $f(x) = 1/x$. As $x \to \infty$, $f(x) \to +0$. The symbol +0 indicates that the graph approaches $y = 0$ from the positive side as $x \to \infty$. One way of checking from which side of an asymptote the graph approaches is to substitute larger and larger (positive or negative) values of x. Here for example $f(x) = 1/x$, $f(1000) = 1/1000 = 0.001 > 0$ so $f(x)$ approaches $y = 0$ from the positive side. Similarly as $x \to -\infty, f(x) \to -0$ then $f(x)$ approaches $y = 0$ from the negative side $x \to -\infty, f(x) \to -0$. The graph of $y = 1/x$ is the rectangular hyperbola.



We may recognise other horizontal asymptotes when $f(x) \to a$ as $x \to \infty$ or $x \to -\infty$.

Example (3)

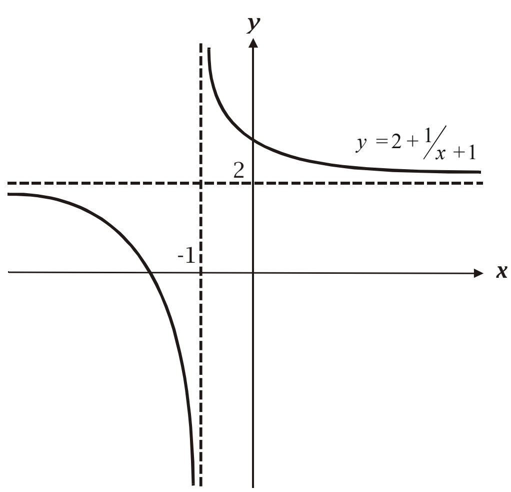
Let $f(x) = 2 + \frac{1}{x+1}$. Given that this function has a vertical asymptote at $x = -1$ sketch its graph.

Solution

We note that $\frac{1}{x+1} \to 0$ as $x \to +\infty$ or $-\infty$ so

As $x \to \infty$, $y \to 2$ from the +ve side

As $x \to -\infty$, $y \to 2$ from the -ve side



Vertical Asymptote

A vertical asymptote occurs for each term

$$\frac{A}{x-\alpha}$$

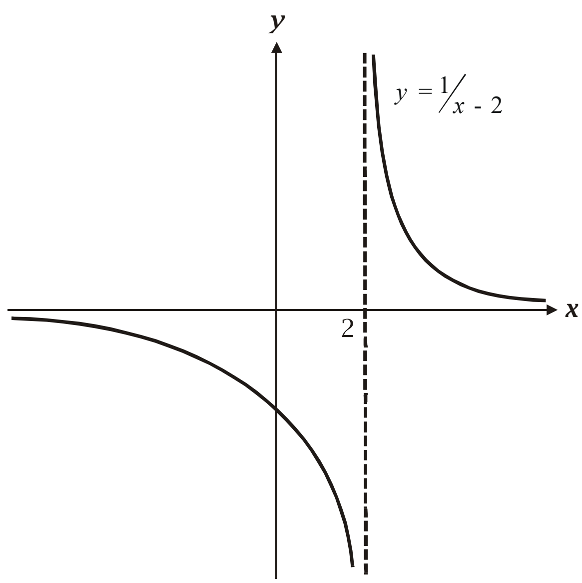
that appears once the rational function has been decomposed into partial fractions. When $x=\alpha$ the denominator of such a fraction becomes 0. It is not possible to divide by 0, hence the function $f(x)$ cannot have a value at $x=\alpha$. The value α is called a *singularity* of the function. As $x \to \alpha$, the graph approaches $x=\alpha$ either by becoming increasingly positive or by becoming increasingly negative. To find out which substitute values of x close to $x=\alpha$.

Example (4)

Sketch the graph of $f(x) = {}^{1}\!/_{x-2}$.

Solution

$f(x) = {}^{1}\!/_{x-2}$ has a vertical asymptote at $x=2$; $f(x)$ is +ve when $x>2$ and -ve when $x<2$.



When there is a term of the form

$(x-\alpha)^2$ or ax^2+bx+c

that cannot be further factorised (in terms of real numbers, because it has a discriminant, $\Delta > 0$) this indicates that the graph approaches the vertical asymptote in the same direction from both sides. The following example illustrates this.

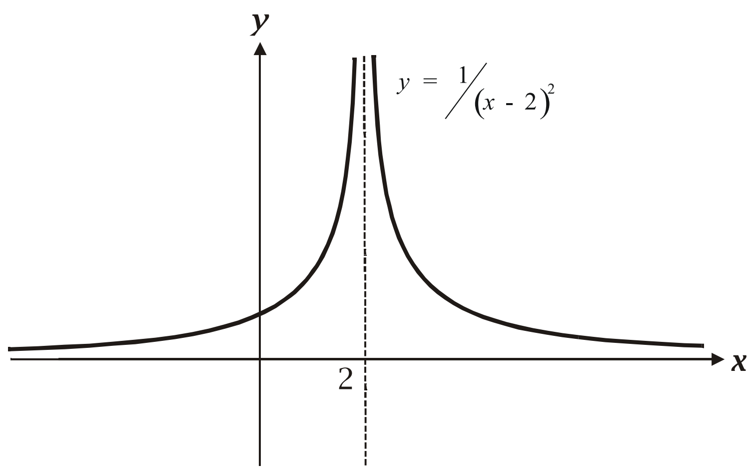
Example (5)

Sketch the graph of $f(x) = \frac{1}{(x-2)^2}$

Solution

$$f(x) = \frac{1}{(x-2)^2}$$

has a vertical asymptote at $x = 2$; $f(x)$ is +ve both when $x > 2$ and when $x < 2$.



Oblique Asymptote

An oblique asymptote occurs when $f(x)$ approaches a linear function of the form

$y = mx + c$

when $x \to +\infty$ and when $x \to -\infty$

Example (6)

Sketch the graph of $f(x) = \frac{x^2 + 3x - 1}{x - 1}$

Solution

Using polynomial division or by inspection

$$y = \frac{x^2 + 3x - 1}{x - 1} = x + 4 + \frac{3}{x - 1}$$

We note that the graph has a vertical asymptote at $x = 1$ and that

$$x \to 1 \qquad \frac{3}{x-1} \to 0$$

Hence

$x \to 1 \qquad f(x) = \dfrac{x^2 + 3x - 1}{x - 1} \to x + 4$

This means that $x + 4$ is an oblique asymptote of $f(x)$. We need to find out which side of this line $f(x)$ approaches as $x \to +\infty$ or $x \to -\infty$. Since $f(x)$ cannot cross the oblique asymptote, it is enough to look at values close to $x = 1$ and determine the character of the vertical asymptote.

As $x \to 1$ from the +ve side $\qquad f(x) \to -\infty$

As $x \to 1$ from the $-$ve side $\qquad f(x) \to +\infty$

so the graph is as follows.