Homomorphisms

Consider S_3 the group of symmetries of the equilateral triangle. We have seen that it has the following combination table.

	I	R_1	R_2	Q_1	Q_2	Q_3
I	I	R_1	R_2	Q_1	Q_2	Q_3
R_1	R_1	R_2	I	Q_2	Q_3	Q_1
R_2	R_2	I	R_1	Q_3	Q_1	Q_2
Q_1	Q_1	Q_2	Q_3	I	R_1	R_2
Q_2	Q_2	Q_3	Q_1	R_1	I	R_1
Q_3	Q_3	Q_1	Q_2	R_2	R_1	I

The partition shows that the group can be divided into four quadrants as follows:

	R	Q
R	R	Q
Q	Q	R

where R stands for any rotation and Q stands for any reflection. The set $\{R, Q\}$ is a group under the operation of composition of transformations. This structure is isomorphic to $\mathbb{Z}_2$ the cyclic group of two elements.

Thus $\mathbb{Z}_2$ is a substructure of S_3 .

In order to capture this concept of one substructure manifested in another we define a homomorphism.

An homomorphism, ϕ, is a mapping (many-one correspondence) such that

1. ϕ: $(G, \text{O}) \to (H, \square)$

2. $\phi(a \text{ O } b) = \phi(a) \square \phi(b)$ for all $a, b \in G$

Thus, the homomorphism property is the same property that preserves structure for isomorphisms. The difference is that isomorphisms are special cases of homomorphisms - they are homomorphisms such that the order of *G* is equal to the order of *H*

$| G| = |H|$

This means that there are exactly the same number of elements in *G* as there are in *H*.

That is, isomorphisms are one-one correspondences (bijections) but homomorphisms are many-one correspondences. Isomorphisms are specials cases of homomorphisms. An isomorphism is a bijective homomorphism.

For S_3 the homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}_2$ maps rotations of S_3 to 0 and reflections of S_3 to 1.

ϕ $\{I, R, R_2\} \rightarrow 0$

$\{Q, Q_2, Q_3\} \rightarrow 1$

The homomorphism property can be verified for any pair of elements of S_3