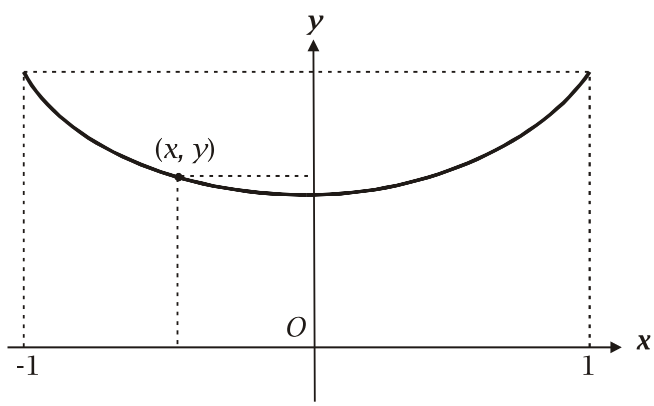
Hyperbolic Functions

The need for hyperbolic functions

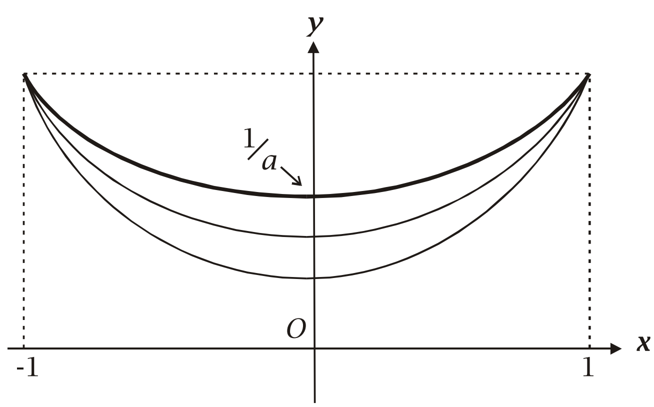
Suppose we hang a rope between two points, which are –1 and +1 units from a fixed point O.



It can be shown that y as a function of x is given by

$$y = \frac{1}{a}\left(\frac{e^{ax} + e^{-ax}}{2}\right)$$

Where a is a constant. The constant a is determined by the length of the rope.



As a increases the curve becomes more bent. It turns out that in applications of the calculus to problems in physics and mechanics the expression $\frac{e^{x} + e^{-x}}{2}$ recurs frequently. It, therefore, is appropriate to define a function that we can equate with this. In fact, we define six *hyperbolic functions* as follows.

Definitions of the hyperbolic functions

Hyperbolic sine of *x*

$$\sinh x \equiv \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

Hyperbolic cosine of *x*

$$\cosh x \equiv \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

Hyperbolic tangent of *x*

$$\tanh x \equiv \frac{\sinh x}{\cosh x} \equiv \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$= \frac{\frac{e^{x}}{e^{-x}} - \frac{e^{-x}}{e^{-x}}}{\frac{e^{x}}{e^{-x}} + \frac{e^{x}}{e^{-x}}}$$

$$= \frac{e^{2x} - 1}{e^{2x} + 1}$$

Hyperbolic secant of *x*

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

Hyperbolic cosecant of *x*

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

Hyperbolic cotangent of *x*

$$\coth x = \frac{1}{\tanh x}$$

Straightforward applications of these definitions are illustrated by the following.

Example (1)

Find the value of x such that $\tanh x = 3/4$.

$$\tanh x = \frac{3}{4}$$

$$\therefore \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{3}{4}$$

$$\therefore 4\left(e^{2x} - 1\right) = 3\left(e^{2x} + 1\right)$$

$$4e^{2x} - 4 = 3e^{2x} + 3$$

$$e^{2x} = 7$$

$$2x = \ln 7$$

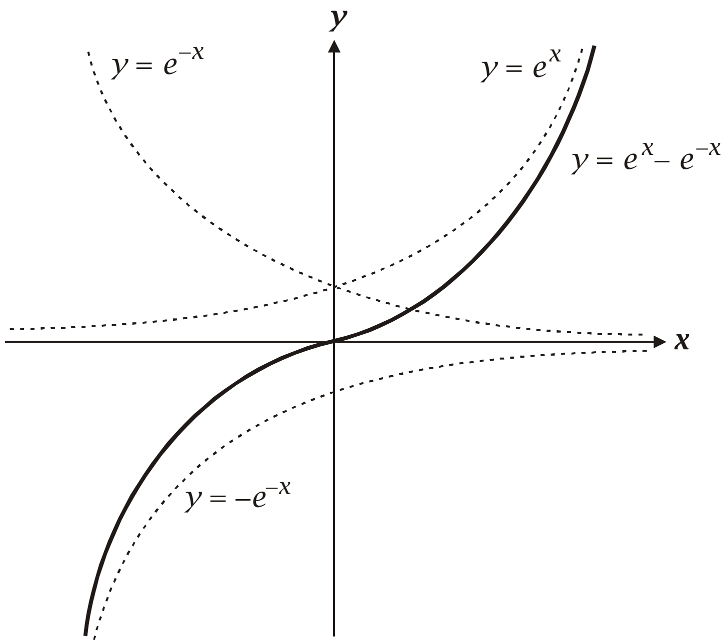
$$x = \frac{1}{2}\ln 7$$

Graphs of the hyperbolic functions

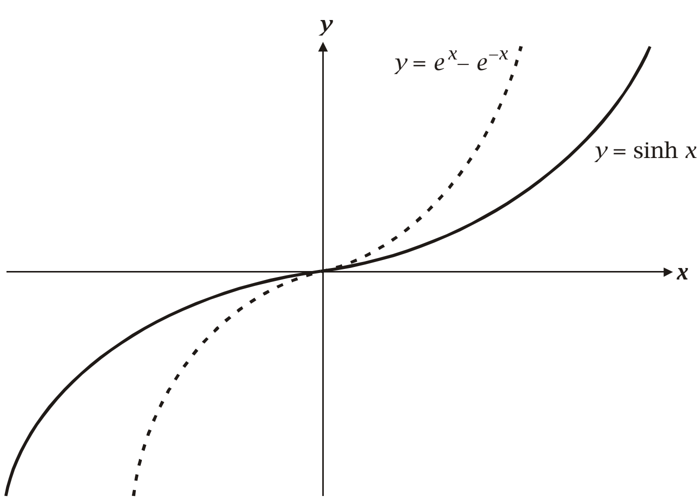
The graphs of these hyperbolic functions are drawn by adding the exponential curves on which they are defined.

Sinh *x*

This following shows the graph $y = 2\sinh x = \left(e^{x} - e^{-x}\right)$



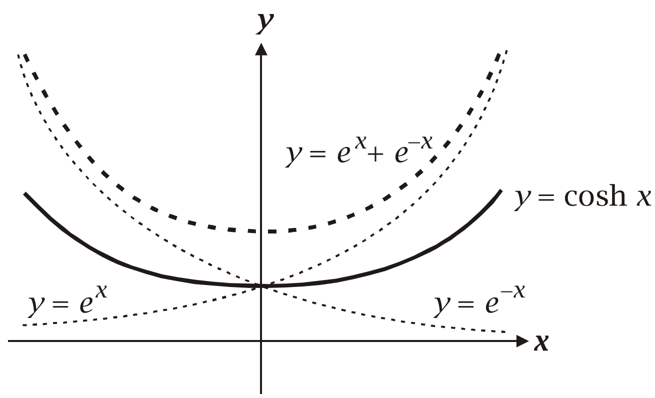
So $y = \sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$ is a scaling of this graph by $\frac{1}{2}$



Example (2)

Sketch the graphs of cosh *x* and tanh *x*.

Cosh *x*



Tanh *x*

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$x \to +\infty \quad \frac{e^{2x} - 1}{e^{2x} + 1} \to \frac{e^{2x}}{e^{2x}} = 1$$

$$x \to -\infty \quad \frac{e^{2x} - 1}{e^{2x} + 1} \to \frac{-1}{1} = -1$$

when $x = 0$ $\qquad \tanh 0 = \frac{1 - 1}{1 + 1} = 0$

It is a monotone increasing function with asymptotes at $y = \pm 1$