# Hyperbolic Identities

## Prerequisites

You should be familiar with the basic trigonometric identities.

**Basic identities**

$$\tan A = \frac{\sin A}{\cos A}$$

$$\sin^2 A + \cos^2 A = 1$$

$$\tan^2 A + 1 = \sec^2 A$$

**Odd and even functions**

$$\sin(-A) = -\sin A$$

$$\cos(-A) = \cos A$$

$$\tan(-A) = -\tan A$$

**Double angle formulae**

$$\sin 2A = 2\sin A \cos A$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\cos^2 A = \frac{\cos 2A + 1}{2}$$

The aim of this chapter will be to demonstrate certain hyperbolic analogues to these identities.

## Hyperbolic identities

**First formula**

The hyperbolic analogue of $\sin^2 A + \cos^2 A = 1$ is

$$\cosh^2 x - \sinh^2 x = 1$$

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Note that in this formula the plus (+) in $\sin^2 A + \cos^2 A = 1$ is replaced by a minus (–) in $\cosh^2 x - \sinh^2 x = 1$

Proof

$\cosh^2 x - \sinh^2 x = (\cosh x - \sinh x)(\cosh x + \sinh x)$

Now $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x})$

$= \frac{1}{2}(e^x - e^x + e^{-x} + e^{-x})$

$= e^{-x}$

$\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})$

$= \frac{1}{2}(e^x + e^x + e^{-x} - e^{-x})$

$= e^x$

$\therefore \cosh^2 x - \sinh^2 x = e^{-x}e^x = e^{x-x} = e^0 = 1$

**Second formula**

The hyperbolic analogue of $\tan^2 A + 1 = \sec^2 A$ is

$1 - \tanh^2 x = \text{sech}^2$

Proof

$\cosh^2 x - \sinh^2 x = 1$

$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$

$1 - \tan^2 x = \text{sech}^2 x$

**Third formulae**

The hyperbolic functions exhibit similar symmetry and anti-symmetry properties to the trigonometric functions. Hyperbolic cosine is an even function; hyperbolic tan and hyperbolic sine are odd functions.

$\cosh(-x) = \cosh x$

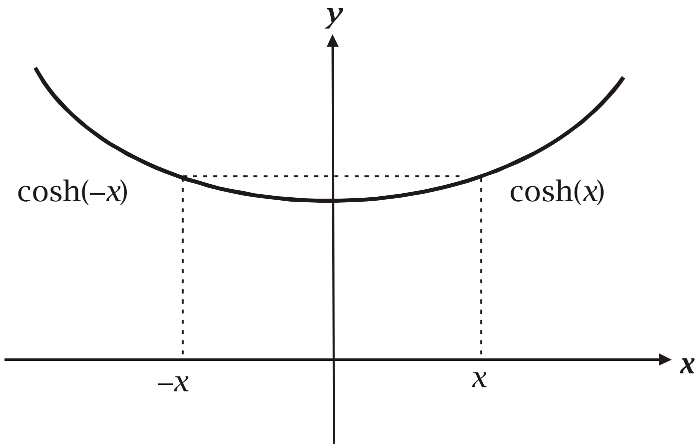
$\sinh(-x) = -\sinh x$

$\tanh(-x) = -\tanh x$

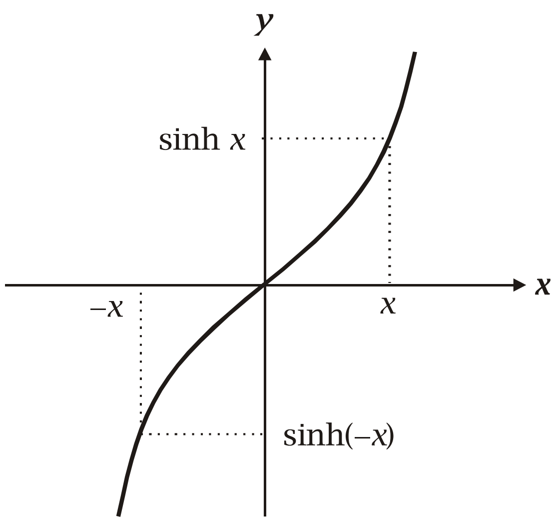
These are proven from their definitions. For example

$\sinh(-x) = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^x - e^{-x}) = -\sinh x$

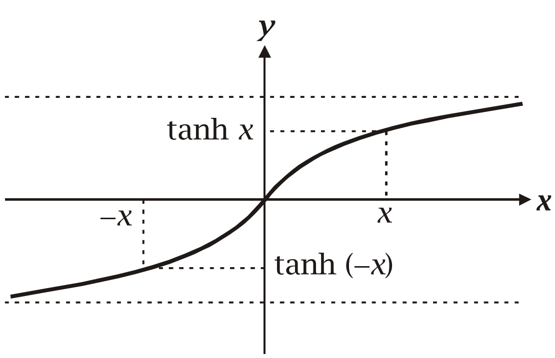
The graphs of the hyperbolic functions illustrate these properties. Cosh $x$ is an even function.




Sinh $x$ is an odd function.




Tanh is also an odd function.

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**Fourth formula**

In the proof of the first formula we have already shown that

$$\cosh x + \sinh x = e^x$$
$$\cosh x - \sinh x = e^{-x}$$

**Fifth formula**

The analogue of the double angle formula of $\sinh x$ is

$$\sinh 2x = 2\sinh x \cosh x$$

Proof

$$\begin{aligned}\sinh 2x &= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) \\ &= \frac{1}{2}\left\{\left(e^{x}\right)^2 - \left(e^{-x}\right)^2\right\} \\ &= \frac{1}{2}\left(e^{x} - e^{-x}\right)\left(e^{x} + e^{-x}\right) \\ &= \frac{1}{2}\sinh x \cosh x\end{aligned}$$

**Sixth formula**

The analogue of the double angle formula for $\cosh x$ is

$$\begin{aligned}\cosh 2x &= \cosh^2 x + \sinh^2 x \\ &= 2\cosh^2 x - 1 \\ &= 1 + 2\sinh^2 x\end{aligned}$$

Proof

$$\begin{aligned}\cosh^2 x + \sinh^2 x &= \left[\frac{1}{2}\left(e^{x} + e^{-x}\right)\right]^2 + \left[\frac{1}{2}\left(e^{x} - e^{-x}\right)\right]^2 \\ &= \frac{1}{4}\left\{\left(e^{x}\right)^2 + 2e^{x}e^{-x} + \left(e^{-x}\right)^2 + \left(e^{x}\right)^2 - 2e^{x}e^{-x} + \left(e^{-x}\right)^2\right\} \\ &= \frac{1}{4}\left\{2e^{2x} + 2e^{-2x}\right\} \\ &= \frac{1}{2}\left(e^{2x} + e^{-2x}\right) \\ &= \cosh 2x\end{aligned}$$

$\therefore \cosh 2x = \cosh^2 x + \sinh^2 x$

But $\cosh^2 x - \sinh^2 x = 1$

so $\cosh^2 x = 1 + \sinh^2 x$ and $\sinh^2 x = 1 + \cosh^2 x$

Hence, $\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

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