Implicit Differentiation

The need for implicit differentiation

It is not always possible to write a function in the form $y = f(x)$. When this can be done the function is said to be defined explicitly. However, the terms *y* and *x* can be so “involved” with each other that *y* cannot be separated from *x*. Such expressions are called “relations” and the variable y is said to be defined implicitly in terms of the variable *x*. For example, the circle has equation $x^2 + y^2 = 1$. We can obtain an equation for *y* in terms of x, but this immediately involves the complication of the introduction of square root.

$$y^2 = \pm\sqrt{1 + x^2}$$

It would be possible to differentiate this function in this form by use of the chain rule, however, it seems that another approach would be useful. This approach is to use *implicit differentiation*. It the expression $x^2 + y^2 = 1$, *y* may be regarded as a function of *x*. The following form makes this clear.

$$y = y(x)$$

The *y* on the right stands for the function as a function of the variable *x*; the *y* on the right stands for the value that the function $y(x)$ takes. Then the expression

$$y^2 = \left[y(x)\right]^2$$

exhibits y^2 as a composite function.

Example (1)

(*a*) Decompose $y^2 = \left[y(x)\right]^2$ into two composite functions.

(*b*) Differentiate

- (i) $y^2 = \left[y(x)\right]^2$
- (ii) x^2
- (iii) 1

(*c*) (i) Differentiate $x^2 + y^2 = 1$ to obtain an expression involving $\frac{dy}{dx}$

(ii) Rearrange this equation to make $\frac{dy}{dx}$ the subject (*d*) (i) Sketch the circle $x^2 + y^2 = 1$

(ii) Mark onto this sketch a tangent at a point (x, y)

(iii) From the sketch write down the gradient at the point (x, y)

Solution

(*a*) Let $f(x) = x^2$ then

$y^2 = f(y) = f(y(x))$

is a composite function

(*b*) (i) By the chain rule

$$\frac{d}{dx}y^2 = 2y \times \frac{dy}{dx}$$

(ii) $\frac{d}{dx}x^2 = 2x$

(iii) $\frac{d}{dx}1 = 0$

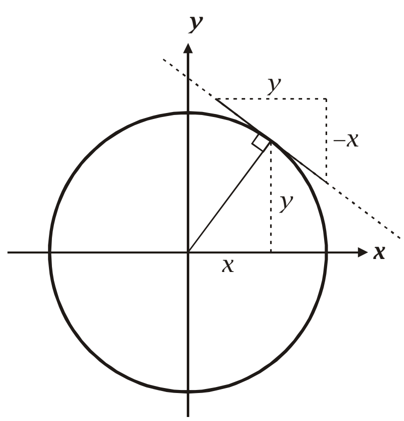
(*c*) (i) $x^2 + y^2 = 1$

$$2x + 2y\frac{dy}{dx} = 0$$

(ii) $2y\frac{dy}{dx} = -2x$

$$\frac{dy}{dx} = -\frac{x}{y}$$

(*d*) (i) and (ii)



(iii) From the sketch we see gradient $= \frac{dy}{dx} = -\frac{x}{y}$

This is implicit differentiation. Effectively, we are applying the chain rule $\frac{dv}{dx} = \frac{dv}{dy} \times \frac{dy}{dx}$ where $v = v(y(x))$ is a composite function to obtain an expression involving $\frac{dy}{dx}$. In the following example we must use both the product and chain rules to find $\frac{dy}{dx}$ implicitly.

Example (2)

Given that $xy^2 = 2x$ find $\frac{dy}{dx}$ in terms of x and y.

Solution

$$xy^2 = 2x$$

$$\frac{d}{dx}\left(xy^2\right) = \frac{d}{dx}(2x)$$

$$\left(\frac{d}{dx}x\right)y^2 + x\left(\frac{d}{dx}y^2\right) = 2 \qquad \text{Product rule}$$

$$y^2 + x \times 2y \times \frac{dy}{dx} = 2 \qquad \text{Chain rule (implicit differentiation)}$$

$$2xy\frac{dy}{dx} = 2 - y^2$$

$$\frac{dy}{dx} = \frac{2 - y^2}{2xy}$$

It is not only polynomial functions that can be differentiated implicitly.

Example (3)

Given that $e^x \sin x = \cos y$ find $\frac{dy}{dx}$ in terms of x and y.

Solution

$$e^x \sin x = \cos y$$

$$e^x \sin x + e^x \cos x = -\sin y \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{e^x\left(\sin x + \cos x\right)}{\sin y}$$

Problem solving involving implicit differentiation

Problems may be set requiring implicit differentiation

Example (4)

Find the equation of the tangent to $x^3 + 2xy + y^2 = 16$ at the point in the first quadrant where $x = 1$.

Solution

$x^3 + 2xy + y^2 = 16$

$3x^2 + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2x + 2y) = -(3x^2 + 2y)$

$\frac{dy}{dx} = -\frac{3x^2 + 2y}{2(x + y)}$

When $x = 1$ we have $1 + 2y + y^2 = 16$

$\therefore y^2 + 2y - 15 = 0$

$(y + 5)(y - 3) = 0$

$\therefore y = -5$ or $y = 3$

The point in the first quadrant has coordinates $(1,3)$

Evaluating $\frac{dy}{dx}$ at the point $(1,3)$

$\left.\frac{dy}{dx}\right|_{(1,3)} = \frac{-3-6}{2+6} = \frac{-9}{8}$

Then $y = {}^{-9}\!/_{8}\,x + c$ passing through $(1,3)$ is the tangent.

Substituting $x = 1,\ y = 3$ into $y = {}^{-9}\!/_{8}\,x + c$ gives

$3 = {}^{-9}\!/_{8}\,x + c \qquad \Rightarrow \qquad c = {}^{33}\!/_{8}$

$\therefore$ the equation of the tangent is $y = {}^{-9}\!/_{8}\,x + {}^{33}\!/_{8}$

Parametric differentiation

A function is a relationship between two variables with general form

$y = f(x)$

However, it is possible that both x and y are given as functions of another variable, called a parameter. An example is

$x(t) = \cos 2t \qquad y(t) = \sin t$

As t varies the point $P = (x(t), y(t))$ moves along the curve whose relationship is $y = f(x)$. The gradient of this curve is given by

$$\frac{dy}{dx} = \frac{{}^{dy}\!/_{dt}}{{}^{dx}\!/_{dt}} = \frac{dy}{dt} \times \frac{dt}{dx}$$

This rule is another application of the chain rule.

Example (5)

Given $x(t) = \cos 2t$, $y(t) = \sin t$ find $\frac{dy}{dx}$ in terms of t.

Solution

$$\frac{dx}{dt} = -2\sin 2t \qquad \frac{dy}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{\cos t}{2\sin 2t}$$

Example (6)

Given that $x(t) = 1 - t^2$, $y(t) = 1 + t^3$, find $\frac{dy}{dx}$ in terms of t.

Solution

$$\frac{dx}{dt} = -2t \qquad \frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{-2t} = -\frac{3}{2}t$$