Implicit differentiation and the second derivative

This material is based on more challenging problems involving the use of implicit and parametric differentiation, and building on the geometric intuition of a good knowledge of conic sections.

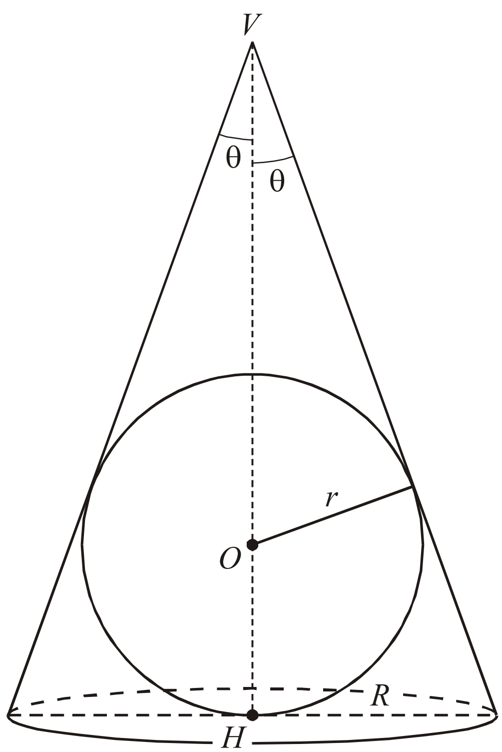
Usually to solve the problem set you have to obtain an expression for the second derivative in cases where the relation between y and x is defined implicitly or parametrically

Example

Find the equation of the right circular cone that is circumscribed about a sphere with radius r and the volume of the cone when it is at a minimum.

Solution

The following diagram illustrates the problem.



Let α be the angle subtended at the top of the cone and let $\theta = \dfrac{\alpha}{2}$.

The strategy for solving this problem is to find the volume as a function of the angle θ, and then to differentiate this function to find when the volume as a minimum. The function will have to be differentiated a second time in order to prove that this is a minimum by the usual criterion that the turning point is a minimum if the second derivative at that point is positive.

Let h be the height of the cone, and let R be the base distance of the cone.

The volume of a cone is given by the usual formula

$$V = \tfrac{1}{3}\pi R^2 h$$

In the diagram the radius of the circle, r, makes a tangent at the point of contact with the cone, so we have a right-angled triangle, and hence

$$OV = \frac{r}{\sin(\theta)}$$

$$h = r + \frac{r}{\sin(\theta)}$$

$$R = h\tan(\theta)$$

Therefore

The volume of hte cone is

$$\begin{aligned} V &= \frac{\pi R^2 h}{3} \\ &= \frac{\pi \left(h\tan(\theta)\right)^2 h}{3} \\ &= \frac{\pi h^3 \tan^2(\theta)}{3} \\ &= \frac{\pi \left(r + \frac{r}{\sin(\theta)}\right)^3 \tan^2(\theta)}{3} \\ &= \frac{\pi r^3 \left(\frac{\sin\theta + 1}{\sin^3\theta}\right)^3 \tan^2\theta}{3} \end{aligned}$$

$$= \frac{\pi r^3}{3} \times \frac{\left(1+\sin(\theta)\right)^3}{\sin^3(\theta)} \times \frac{\sin^2(\theta)}{\cos^2(\theta)}$$

$$= \frac{\pi r^3}{3} \times \frac{\left(1+\sin(\theta)\right)^2 \sin^2\theta}{\sin^3(\theta)\left(1-\sin^2(\theta)\right)}$$

$$= \frac{\pi r^3}{3} \times \frac{\left(1+\sin(\theta)\right)^2}{\sin\theta\left(1-\sin^2(\theta)\right)}$$

Let the function $f(x)$ be such that

$$f(\theta) = \frac{\pi \cdot r^3}{3} \times \frac{\left(1+\sin(\theta)\right)^3}{\sin\theta\left(1-\sin(\theta)\right)}, \quad \text{where} \theta \in \left(0, \frac{\pi}{2}\right)$$

We denote by $\sin\theta = x$. We have $\theta = \sin^{-1}(x)$. From this

$$g(x) = \frac{\pi r^3}{3} \times \frac{(1+x)^2}{x(1-x)}, \quad x \in (0,1).$$

Differentiating g, i.e.

$$g'(x) = \frac{\pi r^3}{3} \times \frac{2(1+x)x(1-x)-(1-x)^2(1-2x)}{x^2(1-x)^2}$$

For turning points

$g'(x) = 0$

That is

$$(1+x)\left(2x(1-x)-(1+x)(1-2x)\right) = 0$$

$$(1+x)\left(2x-2x^2-1+2x-x+2x^2\right) = 0$$

$$(1+x)(3x-1) = 0$$

From this, we get $x = -1$, which is impossible, and $x = \frac{1}{3}$ which is the answer .

To show that this is a minimum, we must differentiate again

$$g''(x) = \frac{\pi r^3}{3}\left(\frac{(1+x)(3x-1)}{x^2(1-x)^2}\right)'$$

$$= \frac{\pi r^3}{3} \times \frac{(3+6x-1)x^2(1-x)^2}{x^4(1-x)^4} - \frac{(1+x)(3x-1)\left[2x(1-x)(1-2x)\right]}{x^4(1-x)^4}$$

$$= \frac{\pi r^3}{3} \times \frac{(6x+2)x^2(x-1)^2 - 2(1+x)(3x-1)x(1-2x)}{x^4(1-x)^4}$$

$$g''\left(\frac{1}{3}\right) = \frac{\pi r^3}{3} \times 4 \times \frac{1}{9} \times \frac{4}{9} \times 3^4 \times \frac{3^4}{2^4} > 0$$

Therefore $x = \frac{1}{3}$ is the minimum point of the function g.

Therefore the angle is $\theta = \sin^{-1}\left(\frac{1}{3}\right)$.

The volume of the cone at this minimum is

$$V\Big|_{\theta=\frac{1}{3}} = \frac{\pi r^3}{3} \times \frac{(1+\sin(\theta))^2}{\sin\theta(1-\sin^2(\theta))} = \frac{\pi r^3}{3} \times \frac{\left(1+\frac{1}{3}\right)^2}{\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^2\right)} = 2\pi r^3$$