Implicit differentiation, further derivatives and MaClaurin series

MaClaurin's series for the expansion of a function $f(x)$ about 0 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots + \frac{f^{(n)}(0)x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

You should have seen applications of this theorem to obtain approximations for specific functions. For example, the Maclaurin series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f^{(3)}(x) = -\cos x$	$f^{(3)}(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = 1$

You will observe that it is not really necessary to continue differentiating this function from "scratch" so to speak. In the case of the second derivative

$$f''(x) = -\sin x = -f(x)$$

Hence, knowing $f(0) = 0$ we obtain automatically $f''(0) = 0$ without recourse to direct substitution into $f''(x) = -\sin x$. This is a labour-saving device, and we might have written the table as follows

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -f(x)$	$f''(0) = 0$
$f^{(3)}(x) = -f'(x)$	$f^{(3)}(0) = -f'(0) = -1$
$f^{(4)}(x) = \frac{d}{dx}\left(-f'(x)\right) = -f''(x) = f(x)$	$f^{(4)}(0) = f(0) = 0$
$f^{(5)}(x) = f'(x)$	$f^{(5)}(0) = f'(0) = 1$

The technique works because at some point a higher derivative is given *implicitly* in terms of lower derivatives. From that point onwards we only need to differentiate these higher derivatives implicitly, without recourse to the original definition of the function.

The value of this technique in short-cutting messy differentiations is best illustrated by worked example.

Example

The function $f(x)$ is defined by $f(x) = e^{-x}\sin x$.

(a) Prove that

$$f''(x) = -2\left(f'(x) + f(x)\right)$$

(b) Differentiate this result further to find $f^{(5)}(0)$, and hence find the series expansion of $f(x)$ up to and including the term in x^5

(c) Estimate by means of this series expansion

$$\int_0^{\frac{1}{2}} e^{-x}\sin x\,dx$$

Give your answer to 3 significant figures

<u>Solution</u>

(a) $f(x) = e^{-x}\sin x$

$$f'(x) = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}\cos x - f(x) \qquad (1)$$

since $f(x) = e^{-x}\sin x$

$$f''(x) = -e^{-x}\cos x + e^{-x}.(-\sin x) - f'(x)$$

$$= -e^{-x}\cos x - e^{-x}\sin x - f'$$

$$= -e^{-x}\cos x - f(x) - f'(x) \qquad (2)$$

Now from equation (1)

$$e^{-x}\cos x = f'(x) + f(x)$$

So substituting this into (2)

$$f''(x) = -\left(f'(x) + f(x)\right) - f(x) - f'(x) = -2\left(f'(x) + f(x)\right)$$

(b) Since

$$f''(x) = -2\left(f'(x) + f(x)\right)$$

$$f^{(3)}(x) = -2\left(f''(x) + f'(x)\right)$$

We can repeat this process indefinitely, and by substituting $x = 0$ obtain the coefficients for the MaClaurin series. Thus

$f(x) = e^{-x}\sin x$ $\quad$ $f(0) = e^{-0}\sin 0 = 1\times 0 = 0$

$f'(x) = e^{-x}\cos x - f(x)$ $\quad$ $f'(0) = e^{-0}\cos 0 - 0 = 1\times 1 - 0 = 1$

$f''(x) = -2\left(f'(x) + f(x)\right)$ $\quad$ $f''(0) = -2(1+0) = -2$

$f^{(3)}(x) = -2\left(f''(x) + f'(x)\right)$ $\quad$ $f^{(3)}(0) = -2(-2+1) = 2$

$f^{(4)}(x) = -2\left(f^{(3)}(x) + f^{(2)}(x)\right)$ $\quad$ $f^{(4)}(0) = -2(2-2) = 0$

$f^{(5)}(x) = -2\left(f^{(4)}(x) + f^{(3)}(x)\right)$ $\quad$ $f^{(5)}(0) = -2(0+2) = -4$

Hence, on substitution into the formula for the MaClaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + ... + \frac{f^{(n)}(0)x^n}{n!} + ...$$

$$f(x) \approx x - \frac{2}{2!}x^2 + \frac{2}{3!}x^3 - \frac{4}{5!}x^5 = x - x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5$$

(c) $$\int_0^{\frac{1}{2}} e^{-x}\sin x\,dx \approx \int_0^{\frac{1}{2}}\left(x - x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5\right)dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^6}{180}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{8} - \frac{1}{24} + \frac{1}{192} - \frac{1}{11520}$$

$$= 0.0885(3.S.F.)$$