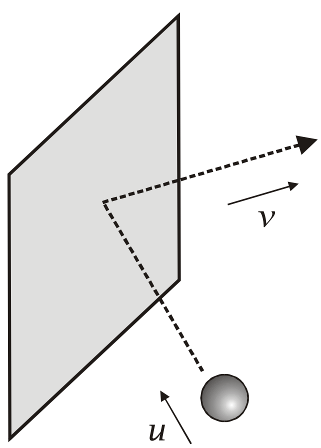
Impulse and elastic collisions in one dimension

Impulse

When two objects collide each exerts a force on the other, the result of which is that one or both cease to move in the same direction as before. Assuming the mass has been unaltered during the collision, and noting that

momentum = mass × velocity $\quad\quad\quad \mathbf{p} = mv$

we conclude that the principle effect of a collision is to change the momentum of one or both of the colliding objects. The following diagram shows a ball colliding with a wall. The effect of the collision is to change the momentum of the ball.



Because of the importance of this concept of change of momentum, we give it a special name of *impulse*, with symbol $\boldsymbol{J}$.

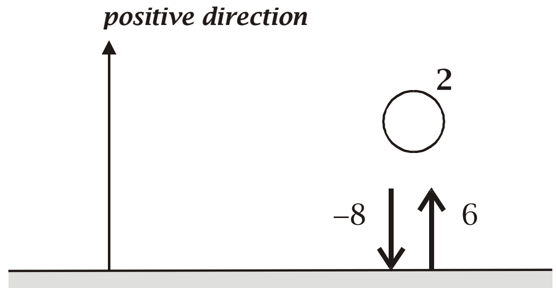
Impulse = change in momentum $\quad\quad\quad \boldsymbol{J} = \Delta\mathbf{p}$

When a collision between two objects causes a change of momentum we say that an impulse has been communicated. Because momentum is a vector, impulse is also a vector. This means that impulse has both a magnitude and a direction. Its units are the same as the units of momentum – that is Newton seconds (Ns).

Example (1)

A ball of mass 2 kg hits the ground with a speed of 8 ms^{-1} and rebounds with a speed of 6 ms^{-1}. Determine the magnitude and direction of the impulse exerted by the ground on the ball.

Solution



Because both velocity and impulse are vectors we must be careful to define from the beginning a positive direction. The diagram above shows how in this question we take the positive direction. The momentum before the collision is

$$\begin{aligned}\mathbf{p}_{before} &= \text{mass } \times \text{velocity} \\ &= 2 \times -8 \\ &= -16 \text{ Ns}\end{aligned}$$

The momentum after the collision is

$$\begin{aligned}\mathbf{p}_{after} &= \text{mass } \times \text{velocity} \\ &= 2 \times 6 \\ &= 12 \text{ Ns}\end{aligned}$$

The change from –16 to 12 Ns is an overall change of 28 Ns. Clearly the impulse is communicated upwards.

We can also derive a formula for this change of momentum, starting with the equation

$$\text{momentum before } + \text{ change in momentum } = \text{momentum after}$$

$$\mathbf{p}_{before} + \Delta\mathbf{p} = \mathbf{p}_{after}$$

Rearrangement gives

$$\text{Impulse} = \Delta\mathbf{p} = \mathbf{p}_{after} - \mathbf{p}_{before}$$

In the last example

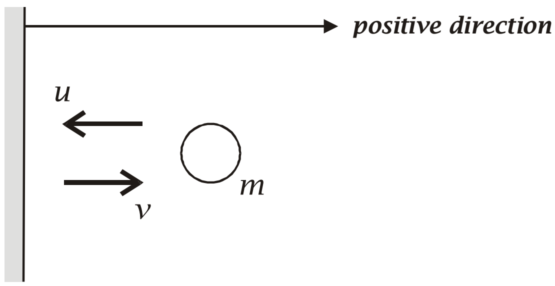
$$\begin{aligned}\boldsymbol{J} &= \mathbf{p}_{after} - \mathbf{p}_{before} \\ &= 12 - (-16) \\ &= 28 \text{ Ns}\end{aligned}$$

where as before the positive value (see diagram) indicates that the impulse is directed upwards and the magnitude of the impulse is 28 Ns.

The equation

$\text{Impulse} = \Delta\mathbf{p} = \mathbf{p}_{after} - \mathbf{p}_{before}$

is of general application. Suppose an object or particle of mass *m* collides with a smooth (that is, frictionless) surface with initial velocity *u* and rebounds with velocity *v*.



Collision of a particle with a perpendicular surface: in this collision the initial velocity, u, *of the particle of mass* m *would be negative because the particle is initially moving in the opposite direction to the way in which the positive axis is defined.*

The impulse imparted to that object by the surface is equal to its change in momentum. Since

$\text{Impulse} = \Delta\mathbf{p} = \mathbf{p}_{after} - \mathbf{p}_{before}$

$J = \mathbf{p}_{after} - \mathbf{p}_{before}$

$J = mv - mu$

Example (2)

A smooth particle of mass 12 kg and travelling with an initial velocity $3\,\text{ms}^{-1}$ collides with a smooth wall as a result of which its velocity is reversed to $-2\ \text{ms}^{-1}$. What is the impulse communicated to the particle?

Solution

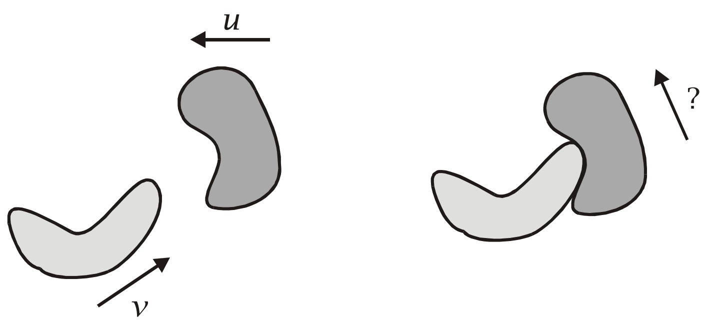
Here the initial velocity is $u = 3\,\text{ms}^{-1}$, the final velocity is $v = -2\,\text{ms}^{-1}$ and the mass is 12 kg. Hence, on substitution

$$J = mv - mu$$
$$= 12 \times -2 - 12 \times 3$$
$$= -60\ \text{Ns}$$

The negative sign indicates that the impulse is communicated in the opposite direction to the initial velocity of the particle.

Smooth collisions

The previous example used the term *smooth*. This term is introduced in order to eliminate complications introduced by rough surfaces that catch on to each other. If one surface acts as a "hook" to the other surface's "handle" then they will lock on to one another regardless of whether they are elastic or not.



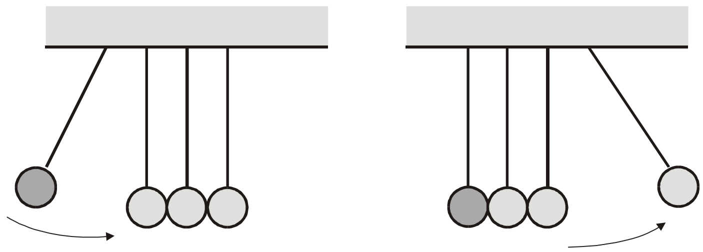
When objects are said to have *smooth* surfaces, this indicates that friction can be ignored. In any collision of two smooth spheres, the impulse of each object on the other is communicated along the line joining the objects' centres of mass.

When these two objects collide the impulse will be communicated along the line joining their centre of mass. This means that they must also rebound along that line.

Elastic collisions and Newton's law of restitution

It is common sense that objects, and substances, differ in their degree of *elasticity*. For example, an "elastic" ball will bounce back up when dropped onto the floor, but a piece of putty will stick to the floor instead. Generally, the bouncy ball will not bounce back to the full height from which it was dropped. If it did bounce back to the full height it would be said to be *perfectly elastic*. For a ball, this would be remarkable – such a ball, once dropped, would carry on bouncing forever. We would not expect this of a ball, because we would expect some of the energy of the ball to be lost with each bounce.

However, some collisions do come close to being perfectly elastic. Hard metal spheres when they collide in a pendulum style arrangement will carry on 'clicking' for a long time.

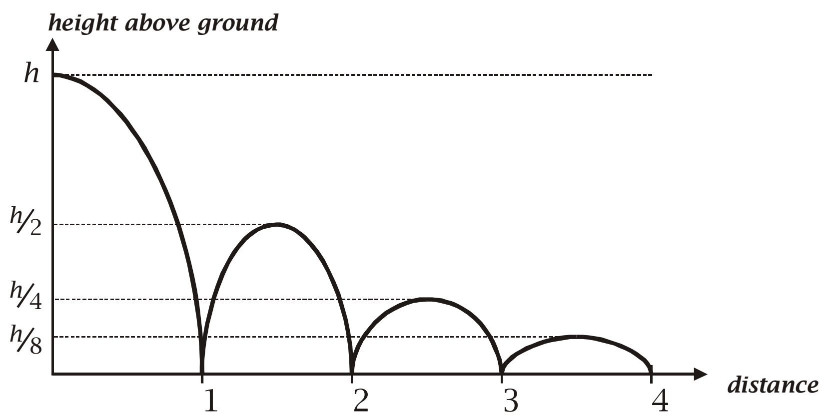


At the other end of the scale, the piece of putty that 'sticks' to the floor is perfectly inelastic. Suppose we drop an elastic (but not perfectly elastic) ball and video the result; suppose also that the ball is bouncing in a vacuum chamber, so we can ignore the effects of air resistance, then it is law of physics that we expect to observe that the height reached by the ball is a constant proportion of the height from which it fell.

Example (3)

A child throws a ball forward with a horizontal velocity of $1\ \text{ms}^{-1}$. The ball also falls under its own weight from a height of *h* metres. Assuming that every time the ball hits the ground it bounces back up to half the height from which it fell, sketch a diagram of its trajectory from the moment it left the child's hand taking the horizontal axis to be the distance from the child and the vertical axis to be the height of the ball above the ground. Show three bounces of the ball.

Solution



Newton discovered that elasticity is a constant property of the contact between smooth surfaces. He defined elasticity in terms of the speed of impact of two objects and their *speed of separation*. The speed of impact is also called the *approach speed*.

We use the term *elasticity* to describe, for example, the response of a spring or elastic band to stretching. Consequently, to distinguish elasticity in that sense from elasticity in the sense being discussed here – the context of collisions – we use the term *coefficient of restitution*. The *coefficient of restitution*, e, is defined as

$$e = \frac{\text{separation speed}}{\text{approach speed}} \qquad e = \frac{v}{u}$$

As already remarked, it is an empirical law that this ratio remains constant for a wide range of situations. This also makes it possible to measure the coefficient of restitution for the collision between objects of given materials in one situation and generalise to other situations. Questions may be set where the coefficient of restitution is known and some aspect of the subsequent trajectory of the particles must be found.

Example (4)

A ball is dropped on to the ground from a height of 1.5m. The coefficient of restitution for the collision is 0.5. Find the height of the first bounce.

Solution

For the speed of impact we use the fact that the gravitational potential energy of the ball is converted to kinetic energy.

$$\Delta K_E = \Delta U$$

$$\frac{1}{2}mu^2 = mgh$$

$$u = \sqrt{2gh}$$

$$u = \sqrt{2 \times 9.81 \times 1.5} = 5.424...ms^{-1}$$

Then Newton's law of restitution

$$\text{separation speed} = e \times \text{approach speed}$$

with $e = 0.5$, gives the speed of separation v as

$$v = 0.5u = 0.5 \times 5.424... = 2.7124...\ ms - 1$$

To find the height

$$\Delta U = \Delta K_E$$

$$mgh = \frac{1}{2}mv^2$$

$$h = \frac{0.5 \times (2.7124...)^2}{9.81} = 0.375 = 0.38 \text{ m} \,(2.s.f.)$$

Example (5)

When a ball is dropped in a vacuum and the trajectory videoed, it is found that the maximum height reached by the ball is always half the height from which it previously fell. Show that the coefficient of restitution is constant for this ball in contact with the surface, and find the value of this coefficient.

Solution

Let h be the height from which the ball is dropped, let u be the impact (approach) speed and v be the separation speed. Then we are required to find the ratio

$$e = \frac{v}{u}$$

and to show that this is constant for each bounce. In the previous example the impact speed, u, is found by equating loss of gravitational potential energy with gain of kinetic energy to be

$$u = \sqrt{2gh}$$

Likewise, the separation speed, v, is found by equating the gain of gravitational energy with the loss of kinetic energy on separation from the surface.

$$\Delta U = \Delta K_E$$

$$mg\left(\frac{h}{2}\right) = \frac{1}{2}mv^2$$

$$v = \sqrt{2g\left(\frac{h}{2}\right)} = \sqrt{gh}$$

$$\therefore e = \frac{v}{u} = \frac{\sqrt{gh}}{\sqrt{2gh}} = \frac{1}{\sqrt{2}} = 0.707 \text{ (3 s.f.)}$$

This coefficient of restitution remains constant since in establishing it we used only the supposition that the height from which the ball was falling was h, which would apply to any height whatsoever.

Elastic collisions in one dimension

We are now ready to tackle a series of examination style questions concerned with elastic collisions in one dimension.

Example (6)

A ball of mass 1.2 kg is thrown vertically downwards with an initial speed of u ms^{-1} from a point 0.8 m vertically above the horizontal ground. It hits the ground with a speed of 8 ms^{-1}.

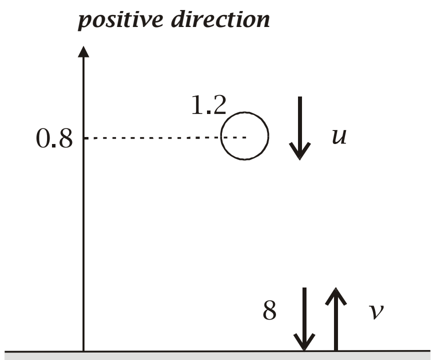
(*a*) Calculate the value of u.

(*b*) Given that coefficient of restitution between the ball and the ground is $\frac{1}{2}$ find the maximum height above the ground to which the ball returns after its first bounce.

(*c*) Determine the magnitude and direction of the impulse exerted by the ground on the ball.

Solution

The following diagram defines what we take to be the positive direction in this question.



Before giving an annotated solution to the problem we shall make an observation. Although the mass is given in the question, it is only necessary to know the mass for the final part of the question when we are asked to calculate the impulse exerted by the ground on the ball. Therefore, in the solution that follows we do not substitute for m in parts (*a*) and (*b*). The reason why the mass is *not* required for parts (*a*) and (*b*) is that in these questions the ball may be treated as a particle moving under gravity, and the acceleration due to gravity, $g = 9.8\,\text{ms}^{-2}$, is independent of mass.

(*a*) As the ball moves towards the ground it gains kinetic energy from its loss of gravitational potential.

gain of kinetic energy = loss of gravitational potential

$$\Delta KE = \Delta U$$
$$\Delta KE = mgh$$

On substituting $h = 0.8$ m this gives

$$\Delta KE = 0.8mg$$

Since the ball strikes the ground with a speed of 8 ms^{-1} its final kinetic energy is

$$\text{final } KE = \frac{1}{2}mv^2 = \frac{1}{2}m \times 8^2 = 32m$$

So its initial kinetic energy must be

$$\text{initial } KE = \text{final } KE - \text{gain in } KE$$
$$= 32m - 0.8mg$$
$$= m(32 - 0.8g)$$

Let the initial velocity of the ball be u ms^{-1}. Then

$$\frac{1}{2}mu^2 = m(32 - 0.8g)$$
$$u^2 = 2(32 - 0.8 \times 9.8) = 48.32$$
$$u = 7.0 \text{ ms}^{-1} \ (2.\text{s.f.})$$

(*b*) $$e = \frac{\text{separation speed}}{\text{approach speed}}$$

$$\text{separation speed} = \frac{1}{2} \times 7.0 = 3.5 \text{ ms}^{-1}$$

To find the maximum height we must substitute into one of the equations of uniform acceleration.

$$v^2 = u^2 + 2as$$
$$0 = 3.5^2 + 2 \times -9.8 \times s$$
$$s = \frac{3.5^2}{2 \times 9.8} = 0.625 = 0.63 \text{ m} \ (2.\text{s.f.})$$

(*c*) The impulse is the change of momentum.

$$\boldsymbol{J} = \Delta\mathbf{p} = mv - mu$$

where u is the velocity before impact and v is the velocity after impact. Hence

$$\boldsymbol{J} = 1.2(3.5 - (-7.0))$$
$$= 1.2 \times 13.5$$
$$= 16.2 \text{ Ns}$$

The direction of motion of the ball has been reversed by the impact, so clearly the impulse is imparted *upwards*. This is indicated by the positive sign. The units of impulse are the same as those of momentum, Ns for Newton seconds.

The next question considers the case where two spheres collide.

Example (7)

Two spheres A and B, of equal radii, lie at rest on a smooth horizontal table. Sphere A has mass 4 kg and sphere B has mass 12 kg. Sphere A is projected with speed 3 ms^{-1} towards sphere B and collides directly with it. The coefficient of restitution between A and B is $\frac{1}{3}$.

(*a*) Find the speeds of A and B after the collision.

(*b*) Determine the magnitude and direction of the impulse exerted by B on A, stating your units.

Solution

(a) Let u be the velocity of A and v be the velocity of B after collision. The following diagram illustrates the solution.

Before collision ***After collision***

In this diagram we have used arrows pointing in the *same* direction for the velocities of the spheres both before and after impact. If in fact a sphere after impact is travelling in the opposite direction, then its velocity will come out as negative. A negative velocity indicates a velocity going in the opposite direction.

Conservation of momentum gives

$$\begin{aligned} &\text{momentum before} = \text{momentum after} \\ &3 \times 4 = 4u + 12v \\ &u + 3v = 3 \qquad\qquad (1) \end{aligned}$$

Newton's law of restitution is

$$e = \frac{\text{separation speed}}{\text{approach speed}}$$

and we are given $e = \frac{1}{4}$. The approach speed is just the speed of A, which is 3ms^{-1}. Regarding the separation speed we need to realise that speed a scalar quantity and may be written using the modulus sign as

$$|v - u| = |u - v|$$

but we want to take the modulus sign off. One of the quantities

$$v - u \text{ or } u - v$$

is positive, and that is the one we want. So we have to be careful to make sure that the values that we substitute into this equation make this equation *positive*. Clearly the velocity of A after impact cannot be larger than that of B, because otherwise A would have overtaken B, which is impossible. So the separation speed in this case is

$$\begin{aligned} \text{separation speed} &= \text{velocity of } B - \text{velocity of } A \\ &= v - u \end{aligned}$$

If in fact the speed u of A is negative, this quantity will still be overall positive. Thus the equation

$$e = \frac{\text{separation speed}}{\text{approach speed}}$$

becomes

$$\frac{1}{3} = \frac{v - u}{3}$$

$$v - u = 1 \qquad (2)$$

To find u and v we must solve the simultaneous equations

$$u + 3v = 3 \qquad (1)$$

$$v - u = 1 \qquad (2)$$

Adding these gives

$$4v = 4 \qquad (1) + (2)$$

$$v = 1 \text{ ms}^{-1}$$

Whence

$$u = 0 \text{ ms}^{-1}$$

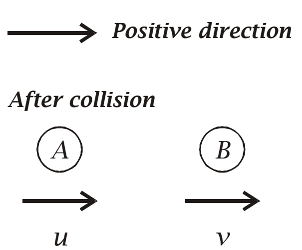
So in fact the impact causes A to cease moving. The momentum is entirely communicated to B.

(b) Since A stops moving after the impact the impulse communicated to B is equal to the momentum of A before the impact. This is

$$\begin{aligned} \boldsymbol{J} &= \text{mass} \times \text{velocity} \\ &= 4 \times 3 \\ &= 12 \text{ Ns} \end{aligned}$$

The direction of the impulse is the same as the original direction of A.

In this question we encountered the problem of how to determine the separation speed of the two particles. The problem may be visualised in the following diagram.



The arrows point in the direction in which we are measuring a *positive* velocity. If in fact one (or both) particles are moving in the opposite direction the *sign* of the velocity will come out negative. So from the point of view of the calculation the direction in which the arrows are pointing is irrelevant, so long as the positive direction is clearly defined. However, looking at the diagram we see that following an impact in which momentum is conserved particle A could never have overtaken particle B, regardless of which direction they are in fact moving. Therefore, with u and v defined as in the diagram the separation speed is *always* given by

$$\begin{aligned}\text{separation speed} &= \text{velocity of } B - \text{velocity of } A \\ &= v - u\end{aligned}$$

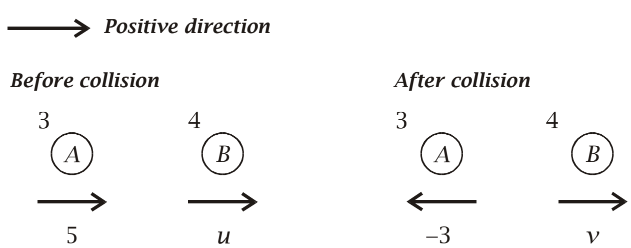
In practice, of course, you may still have to be careful not to get the sign wrong.

Questions of this kind involve two unknown quantities. You are required to substitute into (1) the equation for conservation of momentum, and (2) Newton's law of restitution, to obtain two simultaneous equations in these two unknowns, which may then be solved. An annotated version of the argument, whilst necessary in the first instance, makes the whole process appear more complicated than it is. In the following example we provide only minimal annotation in the solution, since the principle steps should now be clear.

Example (8)

A sphere A, of mass 3 kg, collides directly with another sphere B, of mass 4 kg, on a smooth horizontal surface. Before the collision A moves with speed 5 ms^{-1} and after the collision, it moves with speed 3 ms^{-1} in the opposite direction. The coefficient of restitution between the spheres is $\frac{1}{2}$. Calculate the speeds of B before and after the collision.

Solution



From conservation of momentum

$$\begin{aligned}&\text{momentum before} = \text{momentum after} \\ &3 \times 5 + 4u = 3 \times -3 + 4v \\ &4(v - u) = 24 \\ &v - u = 6 \qquad\qquad (1)\end{aligned}$$

From Newton's law of restitution

$$e = \frac{\text{separation speed}}{\text{approach speed}}$$

$$\frac{1}{2} = \frac{v - (-3)}{5 - u}$$

$$5 - u = 2(v + 3)$$

$$2v + u = -1 \qquad (2)$$

Solving equations (1) and (2) simultaneously

$$3v = 5$$

$$v = \frac{5}{3} \text{ ms}^{-1}$$

$$u = -\frac{13}{3} \text{ ms}^{-1}$$