Impulsive Tensions in Strings

Prerequisites

You should be familiar the definitions of impulse and momentum.

Definition of momentum

momentum = mass $\times$ velocity $\qquad \mathbf{J} = mv$

Principle of conservation of momentum for a collision between two particles

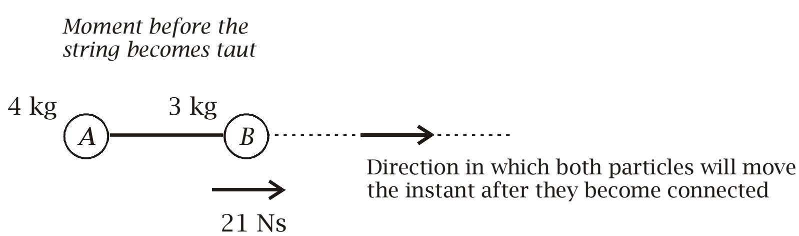
When two particles collide

Total momentum before collision = Total momentum after collision.

The following question essentially involves no new theory but is an application of existing theory to a new situation.

> **Example (1)**
>
> Two particles A and B are connected by means of a light, inextensible string and are initially at rest on a smooth, horizontal surface. Particle A has mass 4 kg and particle B has mass 3 kg. Initially the string is slack when particle B is given an impulse of 21 Ns along the line joining the centres of mass of A and B and in a direction away from A. Find the impulse in the string when the string first becomes taut and the subsequent velocity of both A and B at that instant.

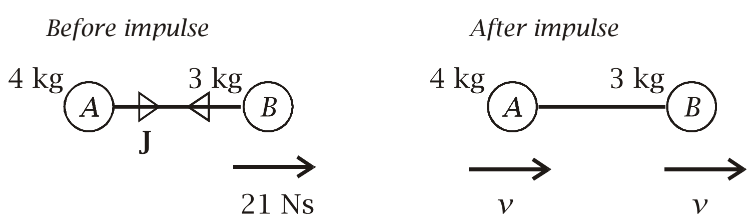
Firstly, let us visualise the problem.



The diagram shows the instant when the string becomes taut. At this instant the two particles become connected. The momentum of B before at that instant is 24 Ns. Part of this momentum will be communicated to A, which is initially at rest. After the impulse is communicated, that is after the jerk of the string, both particles must continue to move in the same direction in which B was moving before the string became taut.

This is similar to a problem involving a collision between two particles A and B where one is initially at rest. A collision similarly communicates an impulse from one object to another. Yet in order to solve a problem involving a collision we would need to know whether any energy was lost in the collision, or alternatively we would require information about the elasticity of the collision. However, here the impulse is communicated from B to A by means of a light, inextensible string, and the surface is smooth, meaning frictionless, so conservation of momentum is sufficient to determine the outcome of the interaction. In a collision one particle can "bounce" off the other and the direction of motion can be reversed. When the impulse is communicated by means of a string, the impulse must be communicated along the string, and in the case we are considering here, both particles must move with the same velocity v, that is, with the same speed and in the same direction.

Solution



The combined mass of the two particles is $M = 4 + 3 = 7$ kg

$$\text{momentum before} = \text{momentum after}$$
$$21 = Mv$$
$$21 = 7v$$
$$v = 3 \text{ ms}^{-1}$$

The impulse is shown in the diagram by the triangle positioned along the string. Two triangles are shown representing the fact that the impulse communicated to A by B is the same as the impulse communicated by B to A. Particle A is accelerated by this impulse from rest, so the impulse is the change in momentum of A given by

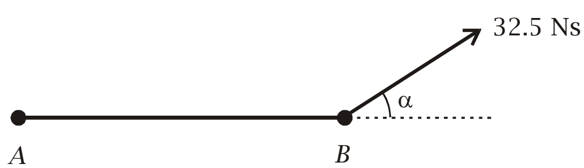
$$\mathbf{J} = \text{change in momentum} = m_A v = 4 \times 3 = 12 \text{ Ns}$$

In two dimensions

In two dimensions the same principles apply. When a string becomes taut an impulse is communicated along the line of the string. If one particle is at rest and is subject to an *impulsive jerk* from a string, then at that instant it must move in the direction in which the impulse was communicated – this being an application of the principle of conservation of momentum.

Example (2)

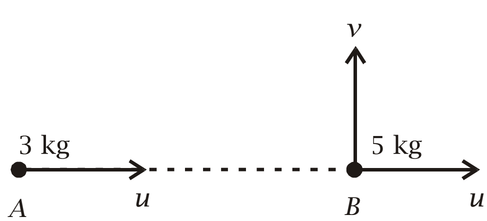
The diagram shows two particles, A with mass 3 kg and B with mass 5 kg, connected by a light inextensible string of length l m. Initially, both particles are lying at rest on a smooth horizontal surface a distance l m apart, with the string just slack. Particle B is given a blow of impulse 32.5 Ns in the direction away from A at an angle α to the line joining A to B where $\tan\alpha = \frac{5}{12}$. Determine the magnitude and direction of the velocities of A and B immediately after the blow.



The solution to this problem rests upon the following intuition. When the particle B is initially struck its momentum (and velocity) have both horizontal and vertical components. Here the impulsive jerk in the string acts only in the horizontal direction. Therefore, the momentum of B in the vertical direction is unaffected by the jerk of the string. This applies only at the **instant** of the jerk.

Solution

Let the horizontal and vertical components of the velocity of B the instant after the string becomes taut be u ms^{-1} and v ms^{-1} respectively. The velocity of A at that instant is also u ms^{-1} in the horizontal direction.

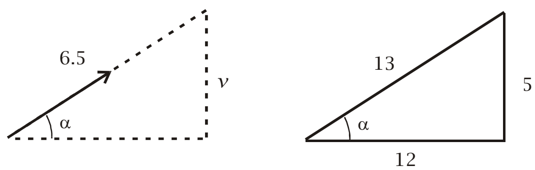


When B was struck it had a velocity of

$$\text{momentum} = \text{mass} \times \text{velocity}$$

$$\text{velocity} = \frac{\text{momentum}}{\text{mass}} = \frac{32.5}{5} = 6.5 \text{ ms}^{-1}$$

The vertical component of the velocity of B after the string becomes taut, which here is v, is the same as vertical component of this initial velocity.



Hence

$$\begin{aligned} v &= 6.5\sin\alpha \\ &= 6.5 \times \frac{5}{13} \\ &= 2.5 \text{ ms}^{-1} \end{aligned}$$

The horizontal component of the momentum given to B when it was struck is

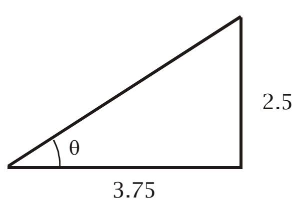
$$\begin{aligned} \mathbf{J} &= 32.5\cos\alpha \\ &= 32.5 \times \frac{12}{13} \\ &= 30 \text{ Ns} \end{aligned}$$

This is equal to the combined momentum of A and B after the string becomes taut.

$$30 = (m_A + m_B)u = 8u$$

$$u = 3.75 \text{ ms}^{-1}$$

The direction of A is along the line joining A to B.



The direction of B is given by θ where

$$\tan\theta = \frac{2.5}{3.75}$$

$$\theta = 33.7° \ (0.1°)$$

and the speed of B is

$$\text{speed } = \sqrt{(3.75)^2 + (2.5)^2} = \sqrt{20.3125} = 4.51 \text{ ms}^{-1} \quad (3 \text{ s.f.})$$

Energy considerations

This sub-section is optional; example (3) is not. In order to understand the effect of an impulsive tension in as string, we shall consider what has happened to the energy in the system of example (2) as a result of the *impulsive tension* in the string when it became taut. The energy before this moment is that of particle B, which was given an impulse of 32.5 Ns, and so was moving with speed

$$\text{speed of } B \text{ before jerk} = \frac{\text{impulse}}{\text{mass}} = \frac{32.5}{5} = 6.5 \text{ ms}^{-1}$$

Hence

$$\text{Kinetic energy of } B \text{ before jerk} = \frac{1}{2}m_B\left(v_B\right)^2 = \frac{1}{2}\times 5\times\left(6.5\right)^2 = 105.625 \text{ J}$$

After the jerk in the string no momentum is lost from the system as a whole. The combined centre of mass of the two particles A and B moves with this impulse and hence

$$\text{speed of centre of mass of } A \text{ and } B \text{ after jerk} = \frac{\text{impulse}}{\text{mass}} = \frac{32.5}{8} = 4.0625 \text{ ms}^{-1}$$

and the energy associated with this is

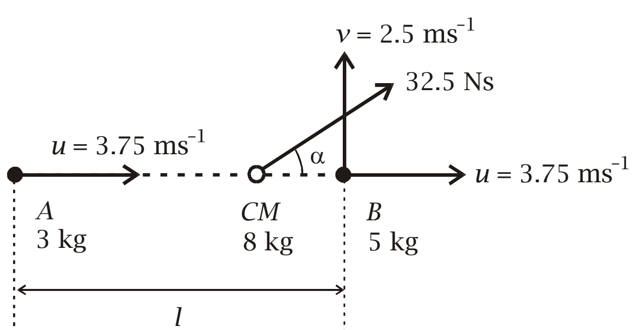
$$\text{Kinetic energy of centre of mass after jerk} = \frac{1}{2}Mv^2 = \frac{1}{2}\times 8\times\left(4.0625\right)^2 = 66.015625 \text{ J}$$

The kinetic energies of A and B after the jerk are

$$\text{Kinetic energy of } A \text{ after jerk} = \frac{1}{2}m_A\left(v_A\right)^2 = \frac{1}{2}\times 3\times\left(3.75\right)^2 = 21.09375 \text{ J}$$

$$\text{Kinetic energy of } B \text{ after jerk} = \frac{1}{2}m_B\left(v_B\right)^2 = \frac{1}{2}\times 5\times\left(4.51...\right)^2 = 50.78125 \text{ J}$$

The total kinetic energy of A and B after the jerk is $50.78125 + 21.09375 = 71.875 \text{ J}$. It appears that the two particles have $71.875 - 66.015625 = 5.859375 \text{ J}$ more linear kinetic energy than that of their combined centre of mass. This is correct. If we examine the following diagram

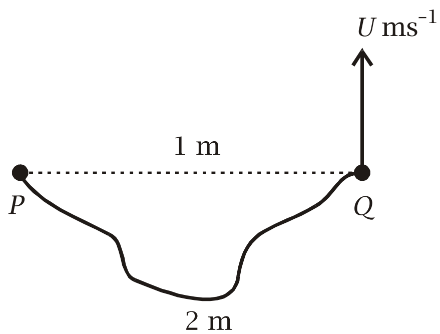


we see that particle B is moving around the centre of mass (CM) and so has rotational kinetic energy as well as linear kinetic energy. At this instant, the horizontal components of the velocity

$\left(u = 3.75\ \text{ms}^{-1}\right)$ do not contribute any rotational momentum or energy. However, the vertical component of the velocity of B $\left(v = 2.5\ \text{ms}^{-1}\right)$ does. We will assert that the missing energy of $5.859375\ \text{J}$ is accounted for by the rotational energy of the entire system about the centre of mass (CM). So after the impulsive jerk the whole system is gyrating about its common centre of mass, whilst the common centre of mass moves off in the direction of the original impulse.[1]

Example (3)

Two particles P and Q of mass 2 kg and 3 kg respectively, are attached one to each end of a light inextensible string of length 2 m. Initially, the particles are at rest on a smooth horizontal surface a distance 1 m apart, as shown in the diagram. Particle Q is then projected horizontally with velocity U ms^{-1} in a direction 90° to the line joining the initial positions of P and Q.



Some time later the string becomes taut. At the instant of the jerk particle Q is found to be moving with a speed of $5\ \text{ms}^{-1}$. Find (*a*) the original velocity of projection of particle

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[1] To show that the rotational energy of A and B (combined) about CM is equal to the missing 5.859375 J is beyond the scope of this chapter. However, for completeness it is given by $E = \frac{1}{2}I\omega^2$ where I is the moment of inertia of the two particles about CM and ω is the angular velocity of this moment of inertia about CM. The centre of mass, CM, is situated $\frac{3}{8}l$ from B and $\frac{5}{8}l$ from A. The moment of inertia of A and B together is

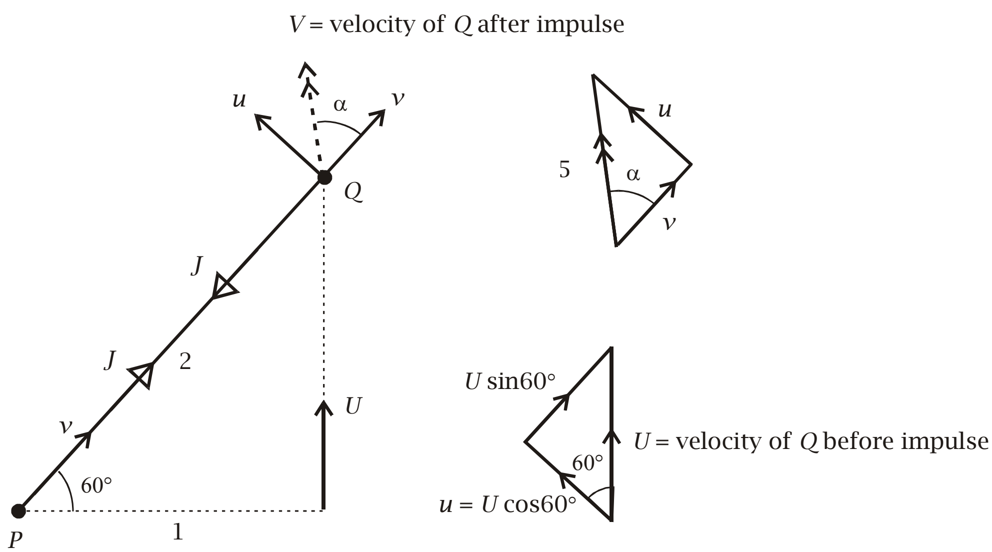
$$I = m_A\left(r_A\right)^2 + m_B\left(r_B\right)^2 = 3\times\left(\frac{5}{3}l\right)^2 + 5\times\left(\frac{3}{8}l^2\right) = \frac{5\times3}{8}l^2$$

Here l is the radius of gyration. The angular velocity of A and B about CM is the same as the angular velocity of B about the radius of gyration $\omega = \frac{v_B}{l} = \frac{2.5}{l}$. Hence the rotational kinetic energy of the system after the jerk is $E = \frac{1}{2}I\omega^2 = \frac{1}{2}\times\left(\frac{3\times5}{8}\right)l^2\times\left(\frac{2.5}{l}\right)^2 = 5.859375\ \text{J}$.

Q, (b) the angle between the velocity of P and the velocity of Q immediately after the jerk, and (c) the impulsive tension in the string during the jerk.

Solution

Let the component of the velocity of Q at the moment of the jerk along the line of the string be v and let the component of the velocity of Q at that same moment perpendicular to the string be u. Let the impulsive tension in the string during the jerk be J. Let $V = 5 \text{ ms}^{-1}$ be the speed of Q after the impulsive jerk.



We are given

$V = \sqrt{v^2 + u^2} = 5 \text{ ms}^{-1}$

$v^2 + u^2 = 25$

Let the angle the string makes with the horizontal at the moment of the jerk be θ. At the instant the string becomes taut the distance between P and Q is 2 m. The horizontal distance is 1 m, so the angle $\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60°$. The jerk in the string does alter the momentum of Q in the direction perpendicular to the string. Hence

$u = U\cos 60° = \frac{U}{2}$

As the diagram indicates because the impulsive tension in the string is conveyed along the line of the string, after the jerk both particles P and Q must be moving with velocity v in the direction of the string. The impulse J causes P of mass 2 kg to accelerate from rest to $v \text{ ms}^{-1}$. Hence

$J = 2v$.

This same impulse causes Q of mass 3 kg to decelerate along the line of the string from $U\sin 60° = \frac{\sqrt{3}}{2}U \text{ ms}^{-1}$ to $v \text{ ms}^{-1}$. Hence

$$J = \frac{\sqrt{3}}{2}U - 3v.$$

Equating expressions for J, we get

$$\frac{\sqrt{3}}{2}U - 3v = 2v$$

$$5v = \frac{\sqrt{3}}{2}U$$

$$v = \frac{\sqrt{3}}{10}U$$

(a) Since $v^2 + u^2 = 25$ we have

$$\left(\frac{\sqrt{3}}{10}U\right)^2 + \left(\frac{U}{2}\right)^2 = 25$$

$$\frac{28}{100}U^2 = 25$$

$$U = 9.4491... = 9.45 \text{ ms}^{-1} \ (3 \text{ s.f.})$$

This is the original velocity of Q before the jerk.

(b) Let α be the angle between the velocities of P and Q after the jerk. Then

$$\tan\alpha = \frac{u}{v} = \frac{\left(\frac{U}{2}\right)}{\left(\frac{\sqrt{3}}{10}U\right)} = \frac{5}{\sqrt{3}}$$

$$\alpha = 70.9° \ (0.1°)$$

(c) $$v = \frac{\sqrt{3}}{10}U = \frac{\sqrt{3}}{10} \times \sqrt{\frac{2500}{28}} = 1.63663...$$

$$J = 2v = 3.27 \text{ Ns} \ (3 \text{ s.f.})$$