Index Equations and Logarithms

Prerequisites

You should understand already (1) the concept of a function and the idea of an inverse function, (2) the distinction between rational and real numbers, including the meaning of an irrational number.

Example (1)

(*a*) State which of these numbers is a rational number and which is an irrational number.

$\sqrt{2}, \sqrt{4}, \sqrt{8}, \sqrt[3]{8}, \sqrt{196}, \sqrt[3]{9}, \sqrt{\dfrac{4}{25}}$

(*b*) The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ stand for sets of numbers. Explain what these sets are.

Solution

(*a*) A rational number is a number that can be expressed as a fraction $\dfrac{p}{q}$ where p and q are whole numbers. An irrational number is a number that is not rational. An irrational number has an infinite decimal expansion. The rule of thumb for recognising irrational numbers is that the root of any prime number is irrational. Hence, $\sqrt{2}$ is irrational. Numbers in the list that are rational are

$\sqrt{4} = 2$
$\sqrt[3]{8} = 2$
$\sqrt{196} = 14$
$\sqrt{\dfrac{4}{25}} = \dfrac{2}{5}$

The other numbers are irrational.

$\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$ and $\sqrt{2}$ is irrational
$\sqrt[3]{9} = \sqrt[3]{3 \times 3} = \sqrt[3]{3} \times \sqrt[3]{3}$ and $\sqrt[3]{3}$ is irrational

(*b*) $\mathbb{N}$ = set of all natural numbers, the counting numbers 0, 1, 2, 3,...

$\mathbb{Z}$ = the set of all integers, positive and negative whole numbers, $\ldots -2, -1, 0, 1, 2, \ldots$

$\mathbb{Q}$ = the set of all rational numbers.

$\mathbb{R}$ = the set of all real numbers that includes all rational and irrational numbers.

The set $\mathbb{R}$ includes all of the set $\mathbb{Q}$ which includes all of the set $\mathbb{Z}$ which includes all of the set $\mathbb{N}$.

The significance of these sets of numbers is that different types of equation are solved by different types of number, i.e. by members of different sets. The solution to the equation $x^2 = 4$ are integers +2 and –2. The solution to the equation $x^2 = \frac{4}{9}$ are rational numbers $\pm\frac{2}{3}$. The solution to $x^2 = 2$ are irrational (real) numbers $\pm\sqrt{2}$. In this chapter we shall be concerned with solutions to equations that will require real numbers.

Example (2)

For the function

$$f\begin{cases}\mathbb{R} \to \mathbb{R} \\ x \to 2x^2 - 1\end{cases}$$

write down

(*a*) The domain and co-domain.

(*b*) Find $f(2), f(-3), f\left(\frac{1}{3}\right)$.

(*c*) The *image* or *range* of a function $y = f(x)$ is the set of all numbers that are possible values for any argument *x* in the domain. Determine the image of the function *f* as defined in this example.

Solution

(a) Domain = $\mathbb{R}$
Codomain = $\mathbb{R}$

(b)

$$f(x) = 2x^2 - 1$$
$$f(2) = 2 \times 2^2 - 1 = 7$$
$$f(-3) = 2 \times (-3)^2 - 1 = 17$$
$$f\left(\frac{1}{3}\right) = 2 \times \left(\frac{1}{3}\right)^2 - 1 = -\frac{7}{9}$$

(c) Any value of the function $f(x) = 2x^2 - 2$ is always a positive number ≥ –1.
Therefore its image set is $x \geq -1$.

Example (3)

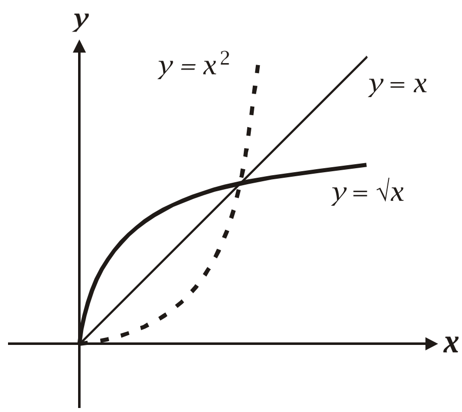
(*a*) What is the inverse of the function $y = f(x) = x^2$?

(*b*) Make a single sketch of the graphs of $y = x^2$ and its inverse and state how the one may be transformed into the other.

Solution

(a) The inverse of the function $y = x^2$ is the function $y = \sqrt{x}$ which is the positive square root of x.

(b) The graph of $y = \sqrt{x}$ is the reflection of the positive part of $y = x^2$ in the line $y = x$.



Graphically the inverse of the function $y = f(x)$ denoted $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$. A final prerequisite of this chapter is that you should be familiar with the rules for manipulating indices.

Summary of the rules for manipulating indices

Definition	$a^n = a \times a \times a \times \ldots \times a \quad (n \text{ times})$
Multiplication	$a^p \times a^q = a^{p+q}$
Division	$a^p \div a^q = a^{p-q}$
Negative Index	$a^{-p} = \dfrac{1}{a^p}$
Zero Index	$a^0 = 1$
Exponent of exponent	$\left(a^p\right)^q = a^{pq}$
Rational Index	$a^{1/p} = \sqrt[p]{a} \qquad a^{p/q} = \left(\sqrt[q]{a}\right)^p = \sqrt[q]{a^p}$

Example (4)

Simplify $\dfrac{x^{\frac{2}{3}} \times \left(x^{-\frac{4}{3}}\right)^2 \times \sqrt[3]{x^2}}{x^{-\frac{4}{3}}}$

Solution

$$\frac{x^{\frac{2}{3}} \times \left(x^{-\frac{4}{3}}\right)^2 \times \sqrt[3]{x^2}}{x^{-\frac{4}{3}}} = \frac{x^{\frac{2}{3}} \times \left(x^{-\frac{4}{3}}\right)^2 \times x^{\frac{2}{3}}}{x^{-\frac{4}{3}}}$$ Rule for a rational index, $\sqrt[a]{x} = x^{\frac{1}{a}}$

$$= \frac{x^{\frac{2}{3}} \times x^{-\frac{8}{3}} \times x^{\frac{2}{3}}}{x^{-\frac{4}{3}}}$$ Rule for exponent of exponent, $\left(a^p\right)^q = a^{p \times q}$

$$= \frac{x^{\frac{2}{3}-\frac{8}{3}+\frac{2}{3}}}{x^{-\frac{4}{3}}}$$ Rule for multiplication, $a^p \times a^q = a^{p+q}$

$$= x^{\frac{2}{3}-\frac{8}{3}+\frac{2}{3}-\left(-\frac{4}{3}\right)}$$ Rule for negative index, i.e. division, $\frac{1}{a^p} = a^{-p}$

$$= x^{\frac{2}{3}-\frac{8}{3}+\frac{2}{3}+\frac{4}{3}}$$

$$= x^0$$

$$= 1$$ Rule for zero index, $a^0 = 1$ for all a

Index equations

The equation $2^x = 32$ is an example of a simple *index equation*. This equation can be solved directly by rewriting 32 as a power of 2.

$2^x = 32$
$2^x = 2^5$
$x = 5$

In the line $2^x = 2^5$ two numbers raised to the same base are equal. Therefore, there indices must also be equal and we are able to infer directly that $x = 5$.

Example (5)

Solve

(a) $3^{x+2} = 27^x$

(b) $2^{2x-5} = 8^{x-4}$

Solution

(a) $3^{x+2} = 27^x$

$3^{x+2} = \left(3^3\right)^x$ $27 = 3^3$

$3^{x+2} = 3^{3x}$ Rule for exponent of an exponent

$x + 2 = 3x$ Equating indices

$x = 1$

(b) $2^{2x-5} = 8^{x-4}$

$2^{2x-5} = \left(2^3\right)^{x-4}$	$8 = 2^3$
$2^{2x-5} = 2^{3(x-4)}$	Exponent of exponent
$2x - 5 = 3(x-4)$	Equating indices
$x = 7$	

Solving index equations by means of a substitution and a problem

The index equation $2^{2x} - 6 \times 2^x + 8 = 0$ cannot be solved by this method. But firstly, observe that

$2^{2x} = \left(2^x\right)^2$

So if we substitute $u = 2^x$ this problem reduces to the problem of solving a quadratic equation.

$2^{2x} - 6 \times 2^x + 8 = 0$
$\left(2^x\right)^2 - 6\left(2^x\right) + 8 = 0$
Substituting $u = 2^x$
$u^2 - 6u + 8 = 0$
$(u-2)(u-4) = 0$
$u = 2$ or $u = 4$
$2^x = 2$ or $2^x = 4$
$x = 1$ or $x = 2$

Example (6)

By means of the substitution $u = 3^x$ solve $3^{2x} - 4 \times 3^x + 3 = 0$

Solution

$3^{2x} - 4 \times 3^x + 3 = 0$
$u^2 - 4u + 3 = 0$
$(u-1)(u-3) = 0$
$u = 1$ or $u = 3$
$3^x = 1$ or $3^x = 3$
$x = 0$ or $x = 1$

However, not all index equations of this type can be solved by this method. When attempting to solve the equation $5^{2x} - 5^x - 6 = 0$ we encounter a difficulty at the last stage.

$5^{2x} - 5^x - 6 = 0$
Substituting $u = 5^x$

$$u^2 - u - 6 = 0$$
$$(u-3)(u+2) = 0$$
$$u = 3 \text{ or } u = -2$$
$$5^x = 3 \text{ or } 5^x = -2$$

The difficulty here is the question of how to solve the equations $5^x = 3$ and $5^x = -2$ in order to obtain the values of *x*. We can see immediately that the second of these equations cannot have a solution because 5^x cannot be a negative number. That leaves $5^x = 3$ as the only valid solution to the original equation, but the problem of how to solve it remains. One possible approach would be to find the numerical solution by use of trial and improvement.

Example (7)

(*i*) Using a calculator find the values of $y = 5^x$ when $x = \frac{2}{3}$ and $x = \frac{5}{6}$.

(*ii*) Let *x* be the solution of index equation $5^x = 3$. Using your answer to part (*i*) state two rational numbers *a* and *b* such that $a < x < b$.

Solution

(i) $y = 5^x$

When $x = \frac{2}{3}$ then $y = 5^{\left(\frac{2}{3}\right)} = 2.9240...$

When $x = \frac{5}{6}$ then $y = 5^{\left(\frac{5}{6}\right)} = 3.8236...$

(ii) Since $2.9240... < 5^x < 3.8236...$

$$\frac{2}{3} < x < \frac{5}{6}$$

So we have discovered an interval within which the solution to $5^x = 3$ lies. That is, the solution *x* to this equation must be a number such that $\frac{2}{3} < x < \frac{5}{6}$. This is progress! However, before taking further steps to solve the problem let us make two observations.

(1) The expression $y = 5^x$ is a *function*.

(2) The solution $\frac{2}{3} < x < \frac{5}{6}$ traps *x* between two *rational* numbers, but the actual solution *x* may not be a rational number. It could be an irrational number lying somewhere between $\frac{2}{3}$ and $\frac{5}{6}$.

The problem we started with was an index equation. The solution requires us (1) to treat the expression $y = 5^x$ as a function and (2) allow this function to have any *real number* value.

Exponential Functions

We need to interpret $y = 5^x$ as a function.

Example (8)

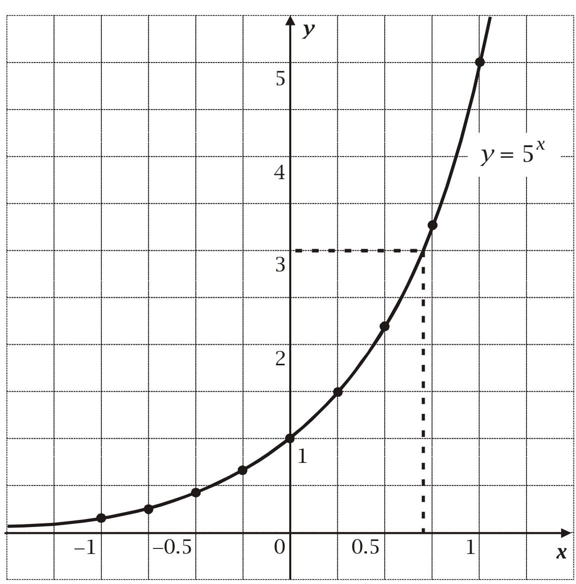
Using a calculator complete the following table for values of $y = 5^x$ giving your answers to 1 decimal place.

x	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y		0.3						3.3	

Plot these points on a graph. Join up the points by means of a smooth continuous curve. Mark on to the graph the value $y = 3$ and indicate the corresponding argument of x for which $5^x = 3$.

Solution

x	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y	0.2	0.3	0.4	0.7	1.0	1.5	2.2	3.3	5.0



The dotted line indicates the value of x such that $5^x = 3$.

By drawing a smooth continuous curve between the points here we allowed the function $y = 5^x$ to take *any* value. So we have made $y = 5^x$ into a real valued function. The solution to $5^x = 3$ is a real number x and we discovered earlier that $\frac{2}{3} < x < \frac{5}{6}$. This has given us an idea of where the solution might lie, but we have not yet *solved* the equation $5^x = 3$.

Example (9)

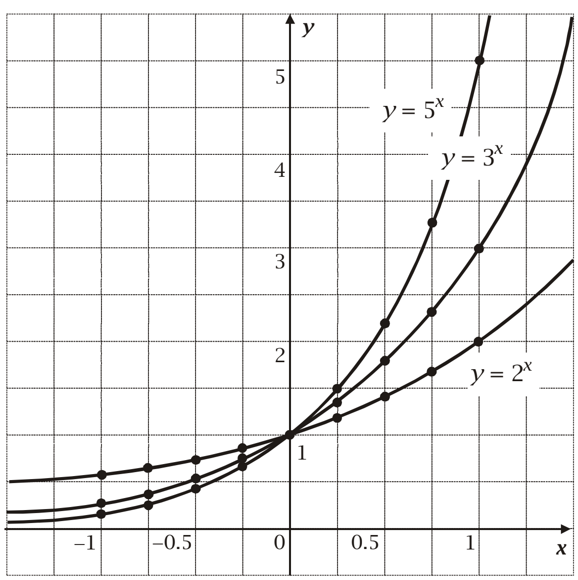
Plot on the same graph the curves of $y = 2^x$, $y = 3^x$ and $y = 5^x$ for values $-1 < x < 1$.

(a) What are the similarities between all these curves?

(b) What are the differences between all these curves?

Solution

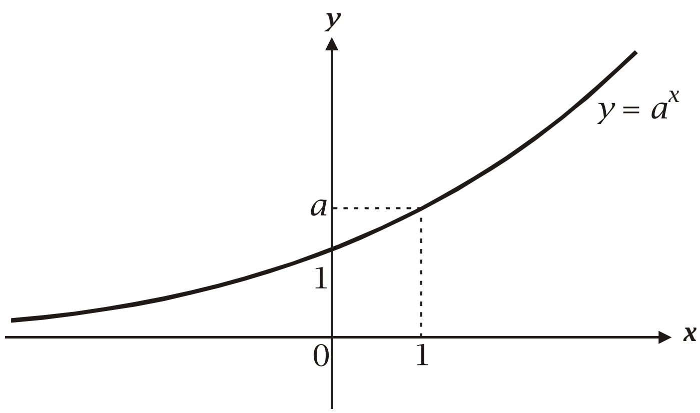
x	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$y = 2^x$	0.5	0.6	0.7	0.8	1.0	1.2	1.4	1.7	2
$y = 3^x$	0.3	0.4	0.6	0.8	1.0	1.3	1.7	2.3	3.0
$y = 5^x$	0.2	0.3	0.4	0.7	1.0	1.5	2.2	3.3	5.0



(a) All the curves have the same basic shape. They are always increasing. The curves are asymptotic to the negative x-axis. They pass through the point $(1, a)$ for $a = 2, 3, 5$. They all pass through the point $(0,1)$.

(*b*) The curves differ from each other only in their degree of steepness: $y = 5^x$ is steeper than $y = 3^x$ which is steeper than $y = 2^x$.

From the solution to this exercise we see that the graph of $y = a^x$ has the same basic shape for all values of a. The number a is called the base and the number x the exponent. Originally, we defined the expression a^x for integer values. We extended the concept of an exponent to cover rational numbers. Now both base and index can be real numbers and the *exponential function* $y = a^x$ is a smooth, continuous ever-increasing curve that is asymptotic to the negative x-axis, and passes through the points $(1, a)$ and $(0, 1)$.



The general graph of the exponential function $y = a^x$

Logarithms

The *logarithm* is the inverse of the exponential function. It is written

$y = \log_a x$

which is read "log to the base a, x". Just as the exponent function depends on two numbers, the base and exponent.

index *or* exponent

a^n

base

so logarithm depends on two numbers, the *base* and the a*rgument*.

argument

$\log_a x$

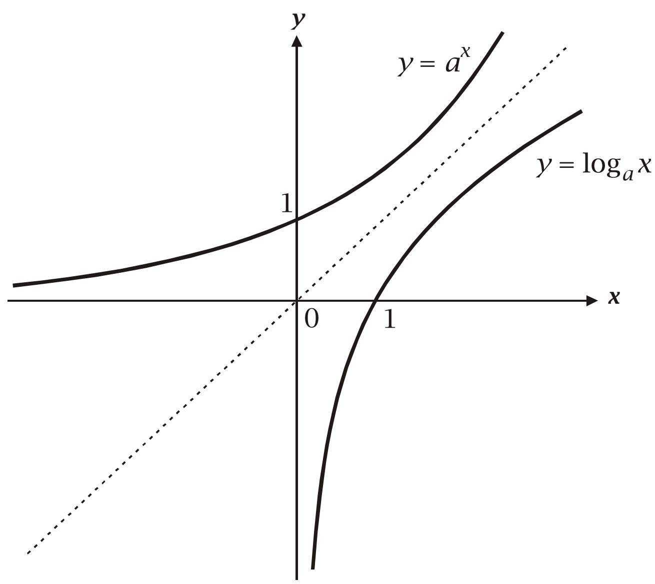
base

The inverse of a function is the reflection of that function in the line $y = x$.

Example (10)

Sketch the inverse (*logarithmic function*) of the exponential function $y = a^x$. Describe the general properties of this function.

Solution



The logarithmic function $y = \log_a x$ like the exponential function $y = a^x$ is an always-increasing function. However, whereas $y = a^x$ gets steeper and steeper, the rate of increase of $y = \log_a x$ gets less and less. Nonetheless, it is always going up. It passes through the point $(1,0)$ on the x –axis, and is asymptotic to the negative *y*-axis. It is undefined for negative values of *x*, so the domain is the positive real line $x > 0$, and 0 is *not* included in the domain.

The point about logarithm being undefined for negative numbers needs to be remembered. If you meet a negative logarithm, for example $y = \log_a(-3)$ in a question, then it is strictly meaningless, and something has gone wrong. Additionally, since all logarithmic graphs pass through the point $(1,0)$ the logarithm of 1 to any base is zero.

$\log_a 1 = 0$ whatever a

This is a consequence of the zero rule for indices $a^0 = 1$. When the base is not written, then the logarithm is understood to be to the base 10.

$\log x = \log_{10} x$.

Rules for the Algebra of Logarithms

(1) Definition of logarithm

The logarithmic function $y = \log_a x$ is the inverse of the exponential function $y = a^x$. The term "logarithm" is often abbreviated to "log". In order to emphasise the relationship between exponential functions and logarithmic functions we sometimes use the symbol exp to denote the exponential function. A mapping diagram is a useful way of understanding the definition of log and its relation to exp.

$$y = a^x \underset{\exp_a}{\overset{\log_a}{\rightleftarrows}} x = \log_a y$$

This shows that if the exponential function, $\exp_a x = a^x$ takes you from one real number x to another y then the logarithmic function, $\log_a$ takes you back again. It "undoes" the exponential function. This definition entails

$\log_a \left(a^x\right) = x$

In this expression log and exp have to be to the same base. If you *change the base* then the inverse relationship *does not hold*. So the following are true

$\log_2 \left(2^x\right) = x$

$\log_3 \left(3^x\right) = x$

But the following where the two base numbers are different is *not true*

$\cancel{\log_3 \left(2^x\right) = x}$

The definition of log entails

$y = a^x \quad \Rightarrow \quad x = \log_a y$

$x = \log_a y \quad \Rightarrow \quad y = a^x$

This really is the starting point for work using logarithms. The arrows tell us how to reverse an expression. Another useful way of remembering these is by using the phrases *taking logs of both sides* and *taking the exponent of both sides*, which you can say in your head as you do so. The meaning of these expressions is illustrated by the following.

$$y = a^x$$
$$\log_a y = \log_a\left(a^x\right) \qquad \text{Taking logs of both sides}$$
$$\log_a y = x \qquad \text{Since } \log_a\left(a^x\right) = x$$
$$x = \log_a y$$

This helps to convert between the two expressions.

Example (11)

Write the following in logarithmic form

(a) $\quad 2^4 = 16$

(b) $\quad 25^{-1/2} = 1/5$

Solution

(a)
$$2^4 = 16$$
$$\log_2\left(2^4\right) = \log_2 16$$
$$4 = \log_2 16$$

(b)
$$25^{-1/2} = 1/5$$
$$\log_{25}\left(25^{-1/2}\right) = \log_{25}\left(1/5\right)$$
$$-1/2 = -\log_{25}\left(1/5\right)$$

(You can step from the first to the third line if you are confident with the definition. If not, put the second line in.)

These examples take you from the index form of an equation to its equivalent logarithmic form. You need to be able to go the other way.

Example (12)

Write the following in index form

(a) $\quad \log_9 3 = \frac{1}{2}$

(b) $\quad \log_2 32 = 5$

Solution

(a)
$$\log_9 3 = \frac{1}{2}$$
$$9^{\log_9 3} = 9^{\frac{1}{2}}$$
$$3 = 9^{\frac{1}{2}}$$

(b) $\log_2 32 = 5$

$32 = 2^5$

To evaluate a logarithm knowledge of the index form of the equation can be useful.

Example (13)

Evaluate the following

(a) $\log_5 125$

(b) $\log_a\left(\frac{1}{\sqrt{a}}\right)$

Solution

(a) $\log_5 125$

Since $5^3 = 125$, then $\log_5 125 = 3$

(b) $\log_a\left(\frac{1}{\sqrt{a}}\right)$

Since $a^{-\frac{1}{2}} = \frac{1}{\sqrt{a}}$ then $\log_a\left(\frac{1}{\sqrt{a}}\right) = -\frac{1}{2}$

To evaluate a logarithm in general see the change of base formula below.

(2) **Addition of logarithms**

The rule for the addition of logarithms is given by

$\log_a b + \log_a c = \log_a bc$

It is essential that the two logarithms are to the same base; if they are not, then nothing follows. The proof of this formula is given in a separate section below.

Example (14)

Simplify $\log 2 + \log 3x$

Solution

In this question no base is indicated. Therefore it is understood that all the numbers are to the base 10.

$$\begin{aligned}\log 2 + \log 3x &= \log(2 \times 3x) \\ &= \log(6x)\end{aligned}$$

This shows that multiplication of numbers with indices to the same base is equivalent to addition of logarithms of those numbers to the same base. Using logarithms we can convert the operation of multiplication into an operation of addition. In the past this was a very useful fact as it enabled large multiplications to be performed by adding logarithms. Nowadays, with the advent of the calculator, this is no longer needed.

(3) **Subtraction of logarithms**

The rule for the subtraction of logarithms to the same base is

$$\log_a b - \log_a c = \log_a \left(\frac{b}{c}\right).$$

The proof of this formula is given later.

Example (15)

Simplify $\log 2 - \log 3x$

Solution

$$\log 2 - \log 3x = \log\left(\frac{2}{3x}\right)$$

(4) **Index rule**

The rule for the index of an index is

$$\left(a^p\right)^q = a^{p\times q}$$

From this we can derive the equivalent rule for logarithms

$$\log_a b^n = n\log_a b$$

We will prove this later.

Example (16)

Simplify $\log 2 + 2\log 3x - \log y$

Solution

$$\begin{aligned}\log 2 + 2\log 3x - \log y &= \log 2 + \log(3x)^2 - \log y\\ &= \log 2 + \log 9x^2 - \log y\\ &= \log\left(2\times 9x^2\right) - \log y\\ &= \log\left(\frac{18x^2}{y}\right)\end{aligned}$$

The rule for indices is applied at the first line with $2\log 3x = \log(3x)^2$.

(5) **Change of Base**

We can find a logarithm to any base give the value of that logarithm to another base. The *change of base* is given by

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Example (17)

Change the following to base 10 and use a calculator to evaluate the result.

(a) $\log_4 9$

(b) $\log_5 2$

Solution

(a) $\log_4 9 = \frac{\log_{10} 9}{\log_{10} 4} = 1.585... = 1.59\ (3.\text{s.f})$

To use a calculator to evaluate $\log_{10} 9$ type $\boxed{\log}\ \boxed{9}$ on the calculator (or possibly $\boxed{9}\ \boxed{\log}$ depending of the type of calculator).

(b) $\log_5 2 = \frac{\log_{10} 2}{\log_{10} 5} = 0.431\ (3.\text{s.f.})$

Index equations revisited

Now we can solve equations by means of logarithms or involving logarithms. Our original problem was to solve the index equation $5^{2x} - 5^x - 6 = 0$. We "got stuck" on the last line.

$5^{2x} - 5^x - 6 = 0$
Substituting $u = 5^x$
$u^2 - u - 6 = 0$
$(u-3)(u+2) = 0$
$u = 3$ or $u = -2$
$5^x = 3$ or $5^x = -2$

We can now proceed by taking the logarithm
$x = \log_5 3$ or $x = \log_5(-2)$

The negative log is undefined so we discard $x = \log_5(-2)$. To solve $x = \log_5 3$ we need to change the base.

$x = \log_5 3 = \frac{\log 3}{\log 5} = 0.68260... = 0.683\ (3.\text{s.f.})$

Example (18)

Solve the equation

$3^{2x-1} = 8$

Solution

$3^{2x-1} = 8$

$\log\left(3^{2x-1}\right) = \log 8$ Taking logs of both sides

$(2x-1)\log 3 = \log 8$ Index rule for logarithms

This is now a linear equation in *x*. The expressions $\log 3$ and $\log 8$ are numbers and can be manipulated as such.

$$(2x-1)\log 3 = \log 8$$
$$2x\log 3 - \log 3 = \log 8$$
$$2x\log 3 = \log 8 + \log 3$$
$$x = \frac{\log 8 + \log 3}{2\log 3}$$
$$= 1.44639...$$
$$= 1.45\ (3.\text{s.f.})$$

You may also be expected to solve an inequality. With inequalities, treat them like equalities with the exception that when multiplying by a negative number reverse the sign of the inequality. To solve an index inequality start just as before by taking logs of both sides.

Example (19)

Solve the following inequality

$5^{-2x} > 8$

Solution

$$5^{-2x} > 8$$
$$\log 5^{-2x} > \log 8$$
$$(-2x)\log 5 > \log 8$$
$$x < \frac{\log 8}{2\log 5}$$
$$x < 0.646014....$$
$$x < 0.646\ \ (3.\text{s.f.})$$

Proofs

If you learn the proofs you will develop a very robust understanding of the topic. This is a fact appreciated by examiners, since they frequently ask you to demonstrate a proof as part of an exam question.

(1) **Addition**

Let $x = \log_a b$ and $y = \log_a c$

Then, $a^x = b$ and $a^y = c$

$\therefore a^x \times a^y = bc$

$a^{x+y} = bc$

$x + y = \log_a bc$

Hence

$\log_a b + \log_a c = \log_a bc$

Exercise (20)

Annotate the above proof indicating how the definition of the logarithm and one particular rule for indices is involved in it.

Solution

Let $x = \log_a b$ and $y = \log_a c$	
Then, $a^x = b$ and $a^y = c$	Definition of logarithm as the inverse of exponent
$\therefore a^x \times a^y = bc$	
$a^{x+y} = bc$	Index rule for multiplication of numbers
$x + y = \log_a bc$	Definition of logarithm as the inverse of exponent
$\log_a b + \log_a c = \log_a bc$	

(2) **Subtraction**

Let $x = \log_a b$ and $y = \log_a c$

Then, $a^x = b$ and $a^y = c$

$\therefore \frac{a^x}{a^y} = \frac{b}{c}$

$a^{x-y} = \frac{b}{c}$

$x - y = \log_a \frac{b}{c}$

Hence, $\log_a b - \log_a c = \log_a \frac{b}{c}$

Exercise (21)

Annotate the proof

Solution

Let $x = \log_a b$ and $y = \log_a c$

Then, $a^x = b$ and $a^y = c$ — Definition of logarithm as the inverse of exponent

$\frac{a^x}{a^y} = \frac{b}{c}$

$a^{x-y} = \frac{b}{c}$ — Index rule for division of numbers to the same base

$x - y = \log_a \frac{b}{c}$ — Definition of logarithm as the inverse of exponent

$\log_a b - \log_a c = \log_a \frac{b}{c}$

(3) **Index**

Let $x = \log_a b$

Then, $a^x = b$

$\therefore \left(a^x\right)^n = b^n$

$a^{nx} = b^n$

$nx = \log_a b^n$

Hence, $n \log_a b = \log_a b^n$

The key step in this proof is the index rule for exponents

$\left(a^p\right)^q = a^{pq}$

(4) **Change of Base**

The change of base is given by

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Proof

Let $x = \log_a b$

Then $a^x = b$

$\therefore \log_c a^x = \log_c b$

$x \log_c a = \log_c b$

$x = \frac{\log_c b}{\log_c a}$

$$\log_a b = \frac{\log_c b}{\log_c a}$$