Indices

Base and index

The symbol a^n means that the number a is multiplied by itself n times. That is

$$a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ times}}$$

For example

$$2^4 = 2 \times 2 \times 2 \times 2$$

The number which is multiplied by itself is called the *base*, and the number of times by which that base is multiplied by itself is called the *index* or *exponent*.

index *or* exponent

$$a^n$$

base

Example (1)

Evaluate $3^3 \times 3^2$.

Solution

$$\begin{aligned} 3^3 \times 3^2 &= (3 \times 3 \times 3) \times (3 \times 3) \\ &= 27 \times 9 \\ &= 243 \end{aligned}$$

Multiplication and division of numbers to the same base

Examine the following

$$2^2 \times 2^3 = (2 \times 2) \times (2 \times 2 \times 2) = 2^5$$

Here we see that the left-hand side of the equation, $2^2 \times 2^3$, results in 2 being multiplied by itself 5 times to give 2^5. The sum of the indices is $2 + 3 = 5$. This illustrates the fact that the multiplication of numbers with the same base follows the rule

$$a^p \times a^q = a^{p+q}$$

That is, when multiplying numbers with the same base you add the indices.

$$2^2 \times 2^3 = 2^{2+3} = 2^5$$

For this to work the numbers must have the same base. It is not permissible to add the indices of numbers that do not have the same base. If this is required, then first the numbers must be converted to the same base.

$4^2 \times 2^2 = 4 \times 4 \times 2^2 = 2^2 \times 2^2 \times 2^2 = 2^{2+2+2} = 2^6$

Example (2)

Solve for x the equation

$9^2 \times 3^3 = 3^x$.

Solution

$$
\begin{aligned}
9^2 \times 3^3 &= 9 \times 9 \times 3^3 \\
&= 3^2 \times 3^2 \times 3^3 \\
&= 3^{2+2+3} \\
&= 3^7
\end{aligned}
$$

Hence $x = 7$.

We also wish to divide numbers to the same base. Examine the following.

$2^3 \div 2^2 = \dfrac{2 \times 2 \times 2}{2 \times 2} = 2^1 = 2$

This illustrates the idea that the division of numbers with the same base follows the rule that you subtract the index of the divisor (number dividing) from the index of the dividend (number divided into).

$a^p \div a^q = \dfrac{a^p}{a^q} = a^{p-q}$

Example (3)

Evaluate $\dfrac{5^5}{5^3 \times 5}$

Solution

Note $5 = 5^1$, so

$$
\begin{aligned}
\frac{5^5}{5^3 \times 5} &= \frac{5^5}{5^3 \times 5^1} \\
&= \frac{5^5}{5^{3+1}} \\
&= \frac{5^5}{5^4} \\
&= 5^{5-4} \\
&= 5^1 \\
&= 5
\end{aligned}
$$

Negative index

Since division of numbers with the same base translates into the subtraction of one index from another, this means that a negative index indicates that division is required, and so denotes a reciprocal.

$a^{-p} = \frac{1}{a^p}$

Hence

$2^{-1} = \frac{1}{2}$ $\quad 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

Example (4)

Write 3×2^{-3} as a fraction.

Solution

$3 \times 2^{-3} = 3 \times \frac{1}{2^3} = \frac{3}{8}$

Zero index

Since we have numbers with negative and positive indices, it makes sense to ask the question, what is the value of a number with a zero index? This value is defined to be $a^0 = 1$ whatever the base, that is whatever *a* is. So

$2^0 = 1$ $\quad 3^0 = 1$ $\quad (2.34)^0 = 1$

To explain why we *define* the zero index in this way, consider the following

$2 \div 2 = 1$

Example (5)

Write $2 \div 2 = 1$ in index form

Solution

$2 \div 2 = 1$ translates into

$2^1 \div 2^1 = 2^{1-1} = 2^0 = 1$

Hence, for consistency, $2^0 = 1$. This argument would apply whatever the base – in other words, it does not depend on the number 2.

Exponent of an exponent

We also want to be able to find the exponent of an exponent. Consider

$(2^2)^3 = (2\times 2)\times(2\times 2)\times(2\times 2) = 2^6$

The product of the two indices in $(2^2)^3$ is 6, and $(2^2)^3$ means 2 multiplied by itself 6 times.

$(2^2)^3 = 2^{2\times 3} = 2^6$

Thus when dealing with the case when a number is raised to an exponent of an exponent,

$(a^p)^q = a^{pq}$

we multiply the exponents.

$(2^2)^3 = 2^{2\times 3} = 2^6$

Example (6)

Simplify $(3a^2bc^3)^2$

Solution

Note that $3a^2bc^3$ can be written

$3a^2bc^3 = 3^1a^2b^1c^3$.

We must multiply each of these exponents by 2.

$$\begin{aligned}(3a^2bc^3)^2 &= (3^1a^2b^1c^3)^2 \\ &= 3^2a^4b^2c^6\end{aligned}$$

Rational index

We also use indices that are rational numbers - that is, numbers that can be written as fractions. Examples of expressions involving an index that is a fraction are

$2^{\frac{1}{2}}$, $5^{\frac{1}{3}}$, $4^{-\frac{1}{5}}$

Example (7)

(i) Evaluate $\sqrt{2}\times\sqrt{2}$

(ii) Use the rule for multiplication of numbers to the same base to find $2^{\frac{1}{2}}\times 2^{\frac{1}{2}}$

(iii) What is the relationship between $\sqrt{2}$ and $2^{\frac{1}{2}}$?

Solution

(i) $\sqrt{2}\times\sqrt{2}=2$

(ii) $2^{\frac{1}{2}}\times 2^{\frac{1}{2}}=2^{\frac{1}{2}+\frac{1}{2}}=2^{1}=2$

(iii) $\sqrt{2}=2^{\frac{1}{2}}$

So the rule for a rational index is

$$a^{1/p}=\sqrt[p]{a}$$

That is, $a^{1/p}$ denotes the pth root of a. For instance

$$8^{1/3}=\sqrt[3]{8}=2$$

Example (8)

Evaluate $\dfrac{1}{(27)^{-\frac{1}{3}}}$

Solution

$$\begin{aligned}\frac{1}{(27)^{-\frac{1}{3}}}&=27^{\frac{1}{3}}\\&=\sqrt[3]{27}\\&=3\end{aligned}$$

Rational indices obey the other rules for multiplication and addition.

Example (9)

Simplify $\dfrac{p^{\frac{1}{2}}\times p^{-\frac{1}{4}}}{p^{-\frac{3}{2}}}$.

Solution

$$\frac{p^{\frac{1}{2}} \times p^{-\frac{1}{4}}}{p^{-\frac{3}{4}}} = \frac{p^{\frac{1}{2}+\left(-\frac{1}{4}\right)}}{p^{-\frac{3}{4}}}$$
$$= \frac{p^{\frac{1}{4}}}{p^{-\frac{3}{4}}}$$
$$= p^{\frac{1}{4}-\left(-\frac{3}{4}\right)}$$
$$= p^{1}$$
$$= p$$

We can also have rational indices that are of the type $\frac{p}{q}$. The rule for these is

$$a^{p/q} = \left(\sqrt[q]{a}\right)^{p} = \sqrt[q]{a^{p}}$$

Example (10)

Evaluate $32^{\frac{2}{5}}$

Solution

$$32^{\frac{2}{5}} = \left(\sqrt[5]{32}\right)^{2}$$
$$= 2^{2}$$
$$= 4$$

Summary

Definition

$a^{n} = a \times a \times a \times \ldots \times a \quad (n \text{ times})$

Multiplication

$a^{p} \times a^{q} = a^{p+q}$

Division

$a^{p} \div a^{q} = a^{p-q}$

Negative Index

$$a^{-p} = \frac{1}{a^p}$$

Zero Index

$$a^0 = 1$$

Exponent of exponent

$$\left(a^p\right)^q = a^{pq}$$

Rational Index

$$a^{1/p} = \sqrt[p]{a}$$

$$a^{p/q} = \left(\sqrt[q]{a}\right)^p = \sqrt[q]{a^p}$$