Induction in Matrix Algebra

Prerequisites

The purpose of this chapter is to extend the technique of mathematical induction to proofs involving matrices. Problems set do not tend to be difficult, but clearly it is essential to be familiar both with mathematical induction and with matrix algebra. Let us remind you, then, that proof by **mathematical induction** is a two-step argument.

Induction Step
If the result is true for the kth number then the result is true for the $(k+1)$th number.
Particular Result
The result is true for $n = 1$ (or for some other starting value).

From which the inference can be drawn:

Conclusion
The result is true for all n (or for all n greater than the starting value).

Regarding **matrix algebra**, a short summary of techniques with which you should be familiar is

Addition of matrices

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a+p & b+q \\ c+d & c+s \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

Multiplication of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathrm{p} & \mathrm{q} \\ \mathrm{r} & \mathrm{s} \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Determinants

$$\det A \text{ or } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

For any $n \times n$ matrix A, and any scalar k: $\det\ (kA) = k \det A$

(By this stage it is assumed that this last result has been demonstrated by algebra for 2 × 2 and 3 × 3 matrices. Here we assume that the result is true for all square matrices, i.e. true for all $n \times n$ matrices. However, to show that would require *mathematical induction*!)

Inverse

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det\ A}\begin{pmatrix} \text{determinants} & \text{determinants} & \text{determinants} \\ \text{derived from} & \text{derived from} & \text{derived from} \\ \text{first cyclic} & \text{second cyclic} & \text{third cyclic} \\ \text{permutatuion} & \text{permutatuion} & \text{permutatuion} \\ \text{of A} & \text{of A} & \text{of A} \end{pmatrix}$$

$$= \frac{1}{\det A}\begin{pmatrix} \text{determinants from} & \text{determinants from} & \text{determinants from} \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix} & \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{\det A}\begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} h & i \\ b & c \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \begin{vmatrix} f & d \\ i & g \end{vmatrix} & \begin{vmatrix} i & g \\ c & a \end{vmatrix} & \begin{vmatrix} c & a \\ f & d \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} g & h \\ a & b \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

$(AB)^{-1} = B^{-1}A^{-1}$

For any $n \times n$ square matrices A, B:

$\det A = \dfrac{1}{\det\left(A^{-1}\right)}$

$\det\left(AB\right) = \det A \det B$

Mathematical induction and matrix algebra

A question typical of those one meets in exams is

Example

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

Prove

$$\mathbf{A}^n = \begin{pmatrix} n+1 & -n \\ n & -n+1 \end{pmatrix}$$

Solution

Particular result

For $n = 1$

$$LHS = \mathbf{A}^1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$RHS = \begin{pmatrix} 1+1 & -1 \\ 1 & -1+1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = LHS$$

So the result holds for $n = 1$

Induction step

Assume true for $n = k$

For $n = k + 1$

$$\mathbf{A}^{k+1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^{k+1}$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^{k}$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k+1 & -k \\ k & -k+1 \end{pmatrix} \qquad \left[\text{By the induction hypothesis}\right]$$

$$= \begin{pmatrix} 2(k+1)-k & -2k+k-1 \\ k+1 & -k \end{pmatrix}$$

$$= \begin{pmatrix} k+2 & -k-1 \\ k+1 & -k \end{pmatrix}$$

$$= \begin{pmatrix} (k+1)+1 & -(k+1) \\ (k+1) & -(k+1)+1 \end{pmatrix}$$

So the induction step holds

Conclusion

Therefore, the result is true for all *n*.