Integrands that integrate to inverse trigonometric functions

The function that integrates to $\sin^{-1} x$

Let $y = \sin^{-1} x$

Then $x = \sin y$

So, since y is a function of x we can differentiate it implicitly with respect to x. Hence

$$\frac{dx}{dx} = \cos y \times \frac{dy}{dx}$$

Since $\frac{dx}{dx} = 1$ that gives us

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

By the trigonometric identity $\sin^2 y + \cos^2 y = 1$

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Hence

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

and so by integrating both sides

$$\int \frac{1}{\sqrt{1 - x^2}}\, dx = \int \frac{dy}{dx} = \sin^{-1} x + c$$

In a similar way we can show that

$$\int -\frac{1}{\sqrt{1 - x^2}}\, dx = \cos^{-1} x + c$$

$$\int \frac{1}{1 + x^2}\, dx = \tan^{-1} x + c$$

Example (1)

Prove $\int \frac{1}{1 + x^2}\, dx = \tan^{-1} x + c$

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Solution

Let $y = \tan^{-1} x$

Then $x = \tan y$

$$\frac{dx}{dx} = \sec^2 y \times \frac{dy}{dx}$$

Since $\frac{dx}{dx} = 1$ that gives us

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

By the trigonometric identity $\sec^2 y = 1 + \tan^2 y$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$\int \frac{1}{1 + x^2}\, dx = \int \frac{dy}{dx} = \tan^{-1} x + c$$

Integrands of the form $\frac{1}{\sqrt{a^2 - x^2}}$

We have already shown that

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

Differentiation of $\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right)$ by the chain rule gives

$$\begin{aligned} \frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) &= \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \times \frac{1}{a} \\ &= \frac{1}{\sqrt{\frac{a^2 - x^2}{a^2}}} \times \frac{1}{a} \\ &= \frac{1}{\frac{1}{a}\sqrt{a^2 - x^2}} \times \frac{1}{a} \\ &= \frac{1}{\sqrt{a^2 - x^2}} \end{aligned}$$

Hence

$$\int \frac{1}{\sqrt{a^2 - x^2}}\; dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

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Integrands of the form $-\frac{1}{\sqrt{a^2 - x^2}}$ and $\frac{1}{a^2 + x^2}$

In a similar way we can show that

$$\int -\frac{1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Example (2)

Prove $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

Solution

$$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{a} \times \left(\frac{1}{1 + \left(\frac{x}{a}\right)^2}\right) \times \frac{1}{a}$$

$$= \frac{1}{a^2} \times \left(\frac{a^2}{a^2 + x^2}\right)$$

$$= \frac{1}{a^2 + x^2}$$

Therefore

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Example (3)

Integrate $\int \frac{1}{\sqrt{16 - x^2}}\,dx$

Solution

This is an application of the formula $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ with $a = 4$

Hence

$$\int \frac{1}{\sqrt{16 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{4}\right) + c$$

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