Integrands that integrate to logarithmic functions

Integrands of the form $\frac{k\,f'(x)}{f(x)}$

Integrals of the form

$$\frac{kf'(x)}{f(x)}\ dx$$

integrate directly to give a logarithmic function of the form

$$\int \frac{kf'(x)}{f(x)}\ dx = k\ln\left|f(x)\right| + c$$

To confirm this result let us differentiate $\ln\left|f(x)\right|$ by the chain rule.

If $f(x) > 0$ $$\frac{d}{dx}\ln\left|f(x)\right| = \frac{d}{dx}\ln\left(f(x)\right) = \frac{1}{f(x)} \times f'(x)$$

and

If $f(x) < 0$ $$\frac{d}{dx}\ln\left|f(x)\right| = \frac{d}{dx}\ln\left(-f(x)\right) = -\frac{1}{f(x)} \times -f'(x) = \frac{f'(x)}{f(x)}.$$

Thus

$$\frac{d}{dx}\ln\left|f(x)\right| = \frac{f'(x)}{f(x)}$$

Since integration is the inverse operation of differentiation

$$\int \frac{f'(x)}{f(x)}\ dx = \ln\left|f(x)\right| + c$$

When multiplied by the constant k this gives

$$\int \frac{kf'(x)}{f(x)}\ dx = k\ln\left|f(x)\right| + c$$

Note that the modulus expression $\left|f(x)\right|$ is required because there is no logarithm of a negative number. However, whether $f(x)$ is positive or negative, it is still the case that

$$\int \frac{kf'(x)}{f(x)}\ dx = k\ln\left|f(x)\right| + c$$

Example (1)

Show that $\int \frac{x}{x^2+1}\ dx = \frac{1}{2}\ln\left|x^2+1\right| + c$

Solution

$$x^2+1>0 \Rightarrow \frac{d}{dx}\left(\frac{1}{2}\ln\left|x^2+1\right|\right) = \frac{d}{dx}\left(\frac{1}{2}\ln\left(x^2+1\right)\right) = \frac{1}{2}\times\frac{1}{x^2+1}\times 2x = \frac{x}{x^2+1}$$

$$x^2+1<0 \Rightarrow \frac{d}{dx}\left(\frac{1}{2}\ln\left|x^2+1\right|\right) = \frac{d}{dx}\left(\frac{1}{2}\ln-\left(x^2+1\right)\right) = \frac{1}{2}\times-\frac{1}{x^2+1}\times -2x = \frac{x}{x^2+1}$$

$$\therefore\ \frac{d}{dx}\left(\frac{1}{2}\ln\left|x^2+1\right|\right) = \frac{x}{x^2+1}$$

$$\therefore \int \frac{x}{x^2+1}\ dx = \frac{1}{2}\ln\left|x^2+1\right| + c$$

Sometimes functions of this type do not appear to be in the form $\frac{kf'(x)}{f(x)}$.

Example (2)

Show that $\int \tan x\ dx = -\ln\left|\cos x\right| + c$

Solution

$$\tan x = \frac{\sin x}{\cos x} = \frac{-(-\sin x)}{\cos x}$$

But $\frac{d}{dx}\cos x = -\sin x$.

So $\tan x$ is strictly of the form $\frac{kf'(x)}{f(x)}$ and

$$\int \tan x\ dx = \int \frac{\sin x}{\cos x}\ dx = -\ln\left|\cos x\right| + c$$

Hence, when integrating trigonometric functions it is sometimes appropriate to write them as quotients.

Example (3)

Find $\int \cot x\ dx$

Solution

$$\cot x = \frac{\cos x}{\sin x} \qquad \Rightarrow \qquad \int \cot x\ dx = \int \frac{\cos x}{\sin x}\ dx = \ln\left|\sin x\right| + c$$