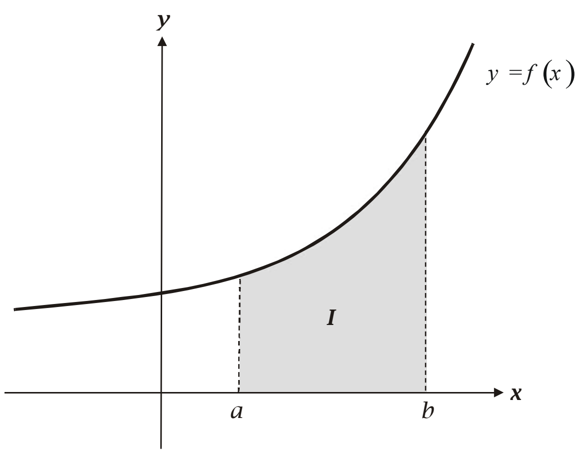
Integration

Areas as definite integrals

The *exact* area under the curve $y = f(x)$ from a to b is denoted by the symbol

$$I = \int_a^b f(x)\,dx$$

and is called the *exact integral* of $y = f(x)$ from a to b. It is read "the integral of the function $f(x)$ from $x = a$ to $x = b$".



Example (1)

The velocity of a car is given by

$$v(t) = 1 + t^2$$

where t is time is seconds. The distance travelled by this car is the area under this curve.

(*a*) Write the distance travelled by the car from $t_1 = 1$ to $t_2 = 4$ as a definite integral of the form $I = \int_a^b f(x)\,dx$.

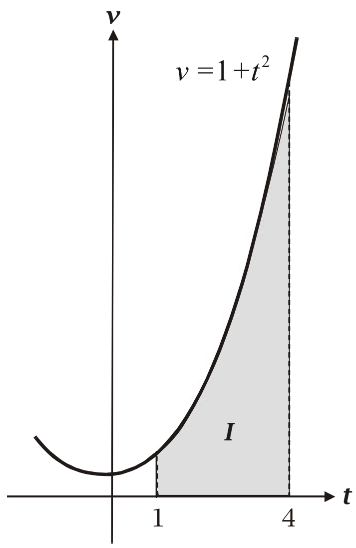
(*b*) Make a sketch of $v(t)$ showing this integral.

Solution

(*a*) Substituting $v(t) = 1 + t^2$ for $f(x)$, dv for dx, $a = 1$ and $b = 4$ into $I = \int_a^b f(x)\,dx$

$$I = \int_1^4 1 + t^2 dt$$

(*b*)



The symbol "dx" in $I = \int_a^b f(x)dx$ is as an essential part of the whole expression. Later we will clarify its meaning.

Example (1) continued

(*c*) Rewrite $I = \int_1^4 1 + t^2 dt$ using the function $y = 1 + x^2$ instead of $v(t) = 1 + t^2$.

Solution

(*c*) $I = \int_1^4 1 + x^2 dx$

Indefinite integrals

A *definite* integral $I = \int_a^b f(x)dx$ has limits written on it – that is numbers a and b specifying the starting and finishing point of the integral. In the example $I = \int_1^4 1 + x^2 dx$ the limits are $a = 1$ and $b = 4$.

An *indefinite integral* takes the form

$g(x) = \int f(x)dx$

In this form there are no limits and $g(x)$ is called the *integral* or *primitive* of $f(x)$. The process of finding the integral of a function $f(x)$ is called *integration*. The integral of $f(x)$ is $\int f(x)dx$.

Integration as the reverse of differentiation –

It can be proven that the process of integration is the reverse of the process of differentiation. If a function $f(x)$ is the derivative of another function $g(x)$ then $g(x)$ is the integral of $f(x)$.

$$g(x) \underset{\text{integrate}}{\overset{\text{differentiate}}{\rightleftarrows}} f(x)$$

This means

$$g(x) = \int f(x)dx$$

and

$$f(x) = g'(x)$$

where $g'(x) = \frac{dx}{dy}$ is the derivative of $g(x)$.

$$g(x) = \int f(x)dx \underset{\text{integrate}}{\overset{\text{differentiate}}{\rightleftarrows}} f(x) = g'(x)$$

Therefore, the primary technique for finding integrals is to reverse the process of finding a derivative.

Example (2)

(*a*) Differentiate $g(x) = x^3 + c$ where c is a constant real number.

(*b*) Using your answer to part (*a*) write down the integral of $f(x) = 2x^3$.

Solution

(*a*) $g(x) = x^3 + c$

$$g'(x) = \frac{dg}{dx} = 3x^2.$$

(*b*) $\int f(x)dx = g(x)$

$\int 3x^2\,dx = x^3 + c$

The term c in this expression is called the *constant of integration*. It stands for an arbitrary real number.

Example (3)

(*a*) Differentiate

$g_1(x) = x^3 + 1$

$g_2(x) = x^3 + 2$

$g_3(x) = x^3 - 2$

(*b*) What is the derivative of any real number constant *c*?

(*c*) Why must the number c be included in the integral $\int 3x^2\,dx = x^3 + c$?

Solution

(*a*) $g_1'(x) = 3x^2$

$g_2'(x) = 3x^2$

$g_3'(x) = 3x^2$

The derivatives of all of these functions are the same.

(*b*) The derivative of a constant c is always zero.

(*c*) Any function of the form $x^3 + c$ has the same derivative, which is $3x^2$. Therefore, the integral of $3x^2$ could be any function $g(x) = x^3 + c$ where c is a real number. So we have to include the letter *c*, denoting this constant to preserve this truth.

As already indicated the term c is called the *constant of integration*. It is usually regarded as an error when integrating a function as an *indefinite integral* to leave it off.

Example (4)

Given that $\frac{d}{dx}x^4 - x^3 = 4x^3 - 3x^2$ find $\int 4x^3 - 3x^2\,dx$.

Solution

Reversing the process of integration and *adding* in the constant of integration we obtain.

$\int 4x^3 - 3x^2\,dx = x^4 - x^3 + c$

The inclusion of the constant of integration c is essential here.

Direct integration

The technique of "direct integration" to find integrals is simply the idea of searching *by trial and error* for a function that reverses the process of differentiation.

Example (5)

Find $\int 2x^3\,dx$

Solution

The derivative of $y = x^4$ is $\frac{dy}{dx} = 4x^3$.

The expression $4x^3$ in $\frac{dy}{dx} = 4x^3$ is twice $2x^3$.

Try $\frac{1}{2}x^4$.

If $y = \frac{1}{2}x^4$ then $\frac{dy}{dx} = 2x^3$

Thus

$\int 2x^3 dx = \frac{1}{2}x^4 + c$

where *c* is the constant of integration.

So direct integration is a method of making educated guesses at what the integral is, then differentiating the guess and improving the guess as a result until the right integral is found. At the last stage the constant of integration must be added. The instruction "integrate $f(x)$ with respect to *x*" means find the indefinite integral $\int f(x)dx$.

Example (6)

Integrate x^5 with respect to *x*.

Solution

Try $\quad x^6 \quad \Rightarrow \quad \frac{d}{dx}x^6 = 6x^5$

This is 6 times bigger than the required answer of x^5.

Try $\quad \frac{1}{6}x^6 \quad \Rightarrow \quad \frac{d}{dx}\frac{1}{6}x^6 = x^5 \quad (\checkmark)$

Therefore $\quad \int x^5 dx = \frac{1}{6}x^6 + c$

Differentiation is easier when functions involving the root symbol or reciprocals are placed in index form.

Example (7)

Integrate $2\sqrt{x}+\dfrac{3}{x^3}$ with respect to *x*.

Solution

Let $f(x)=2\sqrt{x}+\dfrac{3}{x^3}$. Then

$f(x)=2x^{\frac{1}{2}}+3x^{-3}$

We have to integrate two functions. Let us do this separately and then add the results at the end.

For $f_1(x)=2x^{\frac{1}{2}}$

Try $\quad x^{\frac{3}{2}} \quad \Rightarrow \quad \dfrac{d}{dx}x^{\frac{3}{2}}=\dfrac{3}{2}x^{\frac{1}{2}}$

Try $\quad \dfrac{4}{3}x^{\frac{3}{2}} \quad \Rightarrow \quad \dfrac{d}{dx}\left(\dfrac{4}{3}x^{\frac{3}{2}}\right)=\dfrac{4}{3}\times\dfrac{3}{2}x^{\frac{1}{2}}=2x^{\frac{1}{2}} \qquad (\checkmark)$

For $f_2(x)=3x^{-3}$

Try $\quad x^{-2} \quad \Rightarrow \quad \dfrac{d}{dx}x^{-2}=-2x^{-3}$

Try $\quad -\dfrac{3}{2}x^{-2} \quad \Rightarrow \quad \dfrac{d}{dx}-\dfrac{3}{2}x^{-2}=-\dfrac{3}{2}\times-2x^{-3}=3x^{-3} \qquad (\checkmark)$

Thus

$$\int 2\sqrt{x}+\frac{3}{x^3}\,dx=\frac{4}{3}x^{\frac{3}{2}}-\frac{3}{2}x^{-2}+c$$
$$=\frac{4}{3}\sqrt{x^3}-\frac{3}{2x^2}+c$$

Example (8)

Find $\int \sqrt[3]{x}\,dx$

Solution

$f(x)=\sqrt[3]{x}=x^{1/3}$

Try $\quad f(x)=x^{4/3} \quad \Rightarrow \quad f'(x)=\dfrac{dy}{dx}=\frac{4}{3}x^{1/3}$

Try $f(x)=\frac{3}{4}x^{4/3}$ $\Rightarrow$ $f'(x)=\dfrac{dy}{dx}=x^{1/3}$ (✓)

$$I=\int\sqrt[3]{x}\,dx$$
$$=\frac{3}{4}x^{4/3}+c=\frac{3}{4}\sqrt[3]{x^4}+c$$

In a formal solution your trials are not required. These are usually done on a scrap of paper.

Example (9)

Find $\int 4x\left(x^2+3\right)dx$

Solution

$$\int 4x\left(x^2+3\right)dx=\int 4x^3+12x\,dx=x^4+6x^2+c$$

Definite and Indefinite Integrals

We have already explained that the definite integral is the integral of a function with specific limits. It is written

$$I=\int_a^b f(x)\,dx$$

The indefinite integral does not specify the limits

$$F(x)=\int f(x)\,dx$$

In other words, it gives the form of the function which one gets when one integrates $f(x)$, but does not evaluate a specified area. The function $F(x)$, where $F(x)=\int f(x)\,dx$, is called the *integral* or *primitive* of $f(x)$. We have just seen how to find an indefinite integral by the process of direct integration. Whilst the indefinite integral is a family of functions that differ only by the constant of integration, the definite integral is a specific number – for example, it is the specific size of the area under a given curve. The following example will show how a definite integral is evaluated. Later, we will justify the process. The instruction *evaluate* means find the definite number that is equal to the definite integral.

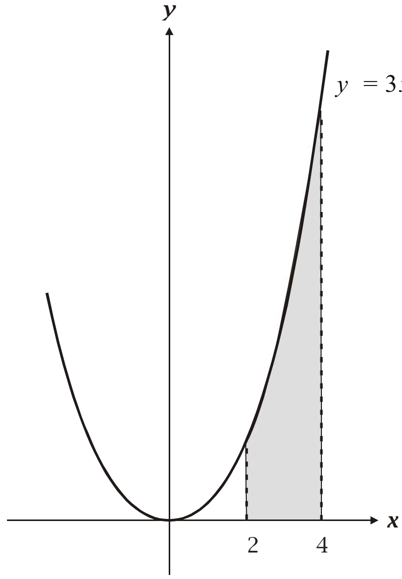
Example (10)

(*a*) Sketch the area corresponding to the definite integral $\int_2^4 3x^2dx$

(*b*) Evaluate $\int_2^4 3x^2dx$

Solution

(*a*)



The instruction evaluate $\int_2^4 3x^2 dx$ is asking you to find the area under the curve $3x^2$ between the points $x = 2$ and $x = 4$.

(*b*) First we find the indefinite integral $\int 3x^2 dx$, which is

$\int 3x^2 dx = x^3 + c$

We add the limits $x = 2$ and $x = 4$ to the indefinite integral as follows.

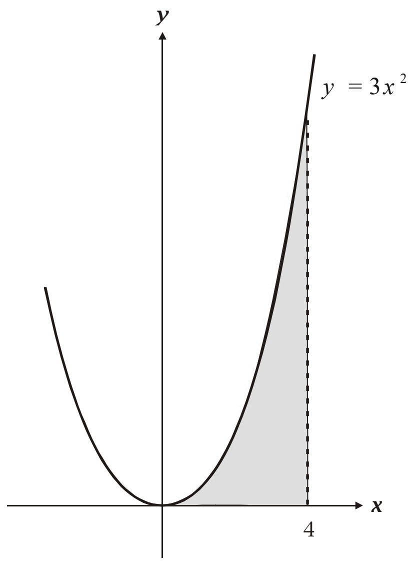
$\int_1^4 3x^2 dx = \left[x^3\right]_2^4$

The integral of $y = 3x^2$, which is $y = x^3$, goes inside large square brackets. The 4 and the 2 are called the *upper* and *lower* limit respectively and have been placed at the top and bottom of the right-hand side of the square brackets respectively. The constant of integration has been *left out*, which is *correct*, as we shall explain below. To *evaluate* the area we substitute the values of the limits into the integral inside the square bracket.

$\int_1^4 3x^2 dx = \left[x^3\right]_2^4 = \left(4^3\right) - \left(2^3\right) = 64 - 8 = 56$ sq. units

In this process we have alternately replaced the x in the x^3 part by 4 and 2, and subtracted the second result from the first to obtain the whole integral.

We need to explain why this works and why there is a minus (subtraction) between the two substitutions. The reason why this process works is because in this example the expression $\left[x^3\right]^4$ gives the area under the curve $y = 3x^2$ from the origin to $x = 4$.



Likewise $\left[x^3\right]_2$ gives the area under $y = 3x^2$ from the origin to x = 2. So when we subtract $\left[x^3\right]_2$ from $\left[x^3\right]^4$ we are obtaining the area under the curve $y = 3x^2$ between x = 2 and x = 4. We could write

$$\int_1^4 3x^2 dx = \left[x^3\right]_1^4$$
$$= \left[x^3\right]^4 - \left[x^3\right]_2$$
$$= \left(4^3\right) - \left(2^3\right) = 64 - 8 = 56$$

The expression $\left[x^3\right]_2^4$ is read "x^3 evaluated between x = 2 and x = 4". The square bracket indicates the process of evaluating. Another symbolism frequently used by mathematicians for precisely the same idea of evaluating is a vertical line. $x^3\Big|_{x=1}^{x=4}$. This is also read "x^3 evaluated between x = 2 and x = 4". It means the same thing.

Example (8)

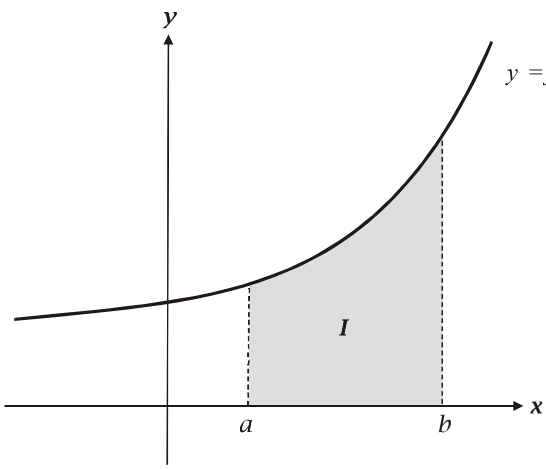
Evaluate $\int_2^5 3x^3 dx$

Solution

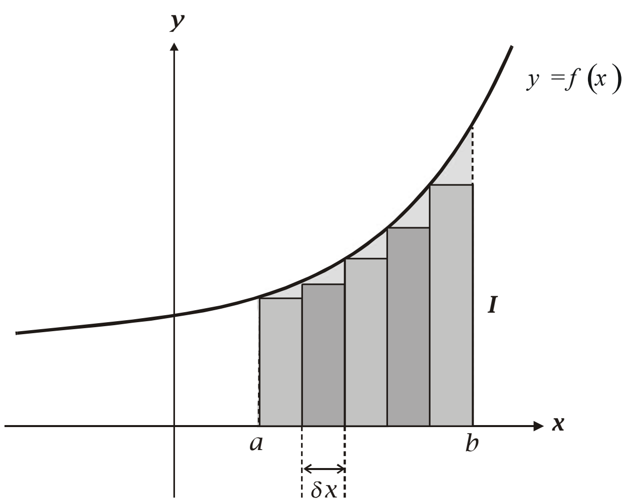
$\int_2^5 3x^3 dx = \left[\frac{3x^4}{4}\right]_2^5$	Find the indefinite integral of $\int 3x^3$. This is $\frac{3x^4}{4}$
$= \left(\frac{3 \times 5^4}{4}\right) - \left(\frac{3 \times 2^4}{4}\right)$	Substitute the limits 5 and 2 for x and subtract
$= \frac{1827}{4}$	Evaluate the fractions
$= 456\frac{3}{4}$	Tidy up

Integration as the Sum of Approximations

This is a theoretical section to help you understand what integration is and you are not directly assessed on this material. If you are already familiar with the trapezium method for approximating areas under curves that will help you understand this section. We are required to find an *approximation* to the area under a given curve, represented by the function $y = f(x)$.



We will approximate the area by rectangles. Each rectangle will have the same width. The width is denoted by δx which means "small increase in *x*".



As the rectangles get smaller and smaller – that is, as the width δx of the rectangle gets smaller – the sum of the area of the rectangles gets closer and closer to the area under the graph.

The area of the $r+1th$ rectangle is

$\delta x \times f(a + r\delta x)$

So the total area of the rectangles is

Area = Sum of n rectangles of size $\delta x \times f(a + r\delta x)$

We will introduce the symbol $\sum$ to mean "sum", and write this as

$A = \sum f(a + r\delta x)\delta x$ for $r = 1$ to $r = n - 1$ (n rectangles in all).

In the limit as δx gets smaller and small and approaches zero this area becomes a better and better approximation to the exact area under the curve. This limit is denoted by $I = \int_a^b f(x)dx$, which we introduced right at the start of this chapter as the expression for the exact area under the curve $y = f(x)$.

$I = \int_a^b f(x)dx \approx A = \sum f(a + r\delta x)\delta x$ for $r = 1$ to $r = n - 1$ (n rectangles in all)

The symbol *dx* indicates that in the expression $I = \int_a^b f(x)dx$ the limit $(n \to \infty)$ has been taken so that the area has become exact. It replaces δx in the expression for the approximate area, which we have seen is

$A = \sum f(a + r\delta x)\delta x$ for $r = 1$ to $r = n - 1$ (n rectangles in all)

The *dx* part of the expression

$$I = \int_a^b f(x)dx$$

is important and without it the symbol

$$\int_a^b f(x)$$

is strictly speaking meaningless, so the *dx* part must be included.