Integration by Parts

Prerequisites

You should be familiar with the product rule for the differentiation of the product of two functions.

Example (1)

(*a*) Write the function $h(x) = x^5 \ln x$ as a product of two functions $h(x) = f(x) \times g(x)$, for appropriate functions *f* and *g*.

(*b*) For the functions found in part (*a*) find $f'(x)$ and $g'(x)$.

(*c*) State the product rule in the form

$(f \times g)' = \ldots$

(*d*) Find $\frac{d}{dx} x^5 \ln x$.

Solution

(*a*) $f(x) = x^5 \qquad g(x) = \ln x$

$h(x) = f(x) \times g(x) = x^5 \ln x$

(*b*) $f'(x) = 5x^4 \qquad g'(x) = \frac{1}{x}$

(*c*) $(f \times g)' = f' \times g + f \times g'$

(*d*) $\frac{d}{dx} x^5 \ln x = 5x^4 \ln x + x^5 \times \frac{1}{x} = x^4 (5 \ln x + 1)$

You should also be familiar with direct integration as a method of finding integrals – that is the process of using *trial and error* to seek for a function which when differentiated gives the function you need to integrate. Trial and error is a laborious method. Imagine trying to find $\int x^4 \ln x \, dx$ by trial and error. Where do you begin? In integrals such as $\int x^4 \ln x \, dx$ *two* functions are multiplied together. *Integration by parts* is a technique that can be used to find the integral of a function which is the product of two other functions.

Integration by parts

Integration by parts is the reverse of the process of differentiation of a product. The product rule for differentiation using the notation of functions is

$$(f \times g)' = f' \times g + f \times g'$$

This can be rearranged to give

$$f \times g' = (f \times g)' - (f' \times g)$$

We can then integrate both sides of this to obtain the formula for *integration by parts*

$$\int fg' = fg - \int f'g$$

This formula is used by replacing an integral that takes the form of the left-hand side by expressions taking the form of the right-hand side. By close examination of the two sides of this equation we see that on the left we have *f* and on the right f' indicating that in replacing the left-hand side by the right-hand side we have to differentiate the function *f*. Likewise, on the left we have g' and on the right *g*, so in using this formula g' must be integrated. So we use integration by parts when the function to be integrated can be written as the product of two functions, one that can be differentiated, the other that can be integrated. The aim is to choose suitable functions for *f* and *g'* such that $\int f' \times g$ is easier to integrate.

Example (2)

Use integration by parts to find $\int x^4 \ln x \, dx$

Solution

Let

$f(x) = \ln x$ $\qquad$ $g'(x) = x^4$

Then

$f'(x) = \frac{1}{x}$ $\qquad$ $g(x) = \frac{1}{5}x^5$

The integration by parts formula is

$$\int fg' = fg - \int f'g$$

Substitution into it gives

$$\int x^4 \ln x \, dx = \frac{1}{5}x^5 \ln x - \int \frac{1}{x} \times \frac{1}{5} \times x^5 dx$$

$$= \frac{1}{5}x^5 \times \ln x - \int \frac{1}{5} \times x^4 dx$$

$$= \frac{1}{5}x^5 \times \ln x - \frac{1}{5}\int x^4 dx$$

Since

$$\int x^4 \, dx = \frac{1}{5}x^5 + c$$

We have

$$\int x^4 \ln x \, dx = \frac{1}{5}x^5 \ln x - \frac{1}{5} \times \frac{1}{5}x^5$$

$$= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5$$

Remark

In $\int x^4 \ln x \, dx$ we clearly recognise $x^4 \ln x$ as a product of two functions. There are two obvious ways in which we can write this product for the purpose of substituting into the parts formula.

First possibility

$f(x) = \ln x \qquad \Rightarrow \qquad f'(x) = \frac{1}{x}$

$g'(x) = x^4 \qquad \Rightarrow \qquad g(x) = \frac{1}{5}x^5$

Second possibility

$f(x) = x^4 \qquad \Rightarrow \qquad f'(x) = 3x^3$

$g'(x) = \ln x \qquad \Rightarrow \qquad ?$

We have already seen that the first possibility leads to a solution. The second possibility leads nowhere! This shows that when using the integration by parts formula the functions f and g' have to be chosen carefully. The aim is to obtain expressions where the problem of integrating the product of functions has been removed.

Example (3)

Find $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} x \sin 3x \, dx$

Solution

$f(x) = x \qquad \Rightarrow \qquad f'(x) = 1$

$g'(x) = \sin(3x) \qquad \Rightarrow \qquad g(x) = -\frac{1}{3}\cos(3x)$

The integration by parts formula is

$$\int fg' = fg - \int f'g$$

Substitution into it gives

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} x\sin 3x\,dx = \left[x\times -\frac{1}{3}\cos(3x)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1\times -\frac{1}{3}\cos(3x)\,dx$$

$$= \left[-\frac{x}{3}\cos(3x)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \frac{1}{9}\left[\sin(3x)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \left(-\frac{\pi}{6}\cos\left(\frac{3\pi}{2}\right) + \frac{\pi}{9}\cos\pi\right) + \frac{1}{9}\left(\sin\left(\frac{3\pi}{2}\right) - \sin\pi\right)$$

$$= -\frac{\pi}{9} - \frac{1}{9}$$

$$= -\left(\frac{\pi+1}{9}\right)$$