Integration of Rational Functions by Decomposition into Partial Fractions

Integrating partial fractions

The technique of decomposition of rational functions into partial fractions is used to bring rational functions into a form in which they can be integrated. When integrating partial fractions one has to recall the following standard integrals

$$\int \frac{1}{x}\ dx = \ln(x) + c$$

$$\int \frac{f'(x)}{f(x)}\ dx = \ln|f(x)| + c$$

$$\int \frac{1}{1+x^2}\ dx = \tan^{-1} x + c$$

$$\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

When a rational function is decomposed into its partial fractions, the resultant fractions are often of the form

$$\frac{A}{x+\alpha} \qquad \frac{Ax}{x^2+\alpha^2} \qquad \frac{1}{x^2+\alpha^2}$$

Although expressions of the form $\frac{1}{\sqrt{a^2-x^2}}$ do not occur in the context of partial fractions, it is also useful to bear in mind also the following results.

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + c \qquad \int \frac{1}{\sqrt{a^2-x^2}}\ dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{-1}{\sqrt{1-x^2}}\ dx = \cos^{-1} x + c \qquad \int -\frac{1}{\sqrt{a^2-x^2}}\, dx = \cos^{-1}\left(\frac{x}{a}\right) + c$$

Example (1)

Express $\frac{5}{(x+1)(x^2+4)}$ in the form $\frac{A}{x+1} + \frac{Bx}{x^2+4} + \frac{C}{x^2+4}$ and hence show that

$$\int_0^2 \frac{5}{(x+1)(x^2+4)}\ dx = \frac{1}{2}\ln\left(\frac{9}{2}\right) + \frac{1}{8}\pi$$

Solution

Let

$$\frac{5}{(x+1)(x^2+4)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+4} \equiv \frac{A(x^2+4)+(Bx+C)(x+1)}{(x+1)(x^2+4)}$$

$$\therefore 5 \equiv A(x^2+4)+(Bx+C)(x+1)$$

$x=-1 \;\Rightarrow\; 5=5A \;\Rightarrow\; A=1$
$x=0 \;\Rightarrow\; A=1 \;\Rightarrow\; 5=4+C \;\Rightarrow\; C=1$
$x=1 \;\Rightarrow\; A=1 \;\Rightarrow\; C=1 \;\Rightarrow\; 5=5+2(B+1) \;\Rightarrow\; 2(B+1)=0 \;\Rightarrow\; B=-1$

$$\therefore \frac{5}{(x+1)(x^2+4)} = \frac{1}{x+1} - \frac{x}{x^2+4} + \frac{1}{x^2+4}$$

$$\begin{aligned}
\int_0^2 \frac{5}{(x+1)(x^2+4)}\,dx &= \int_0^2 \frac{1}{x+1}\,dx - \int_0^2 \frac{x}{x^2+4}\,dx + \int_0^2 \frac{1}{x^2+4}\,dx \\
&= \Big[\ln|x+1|\Big]_0^2 - \left[\frac{1}{2}\ln|x^2+4|\right]_0^2 + \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 \\
&= (\ln 3 - \ln 1) - \left(\frac{1}{2}\ln 8 - \frac{1}{2}\ln 4\right) + \left(\frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0\right) \\
&= \frac{1}{2}(\ln 9 - \ln 8 + \ln 4) + \frac{1}{2}\times\frac{\pi}{4} \\
&= \frac{1}{2}\ln\left(\frac{9\times 4}{8}\right) + \frac{\pi}{8} \\
&= \frac{1}{2}\ln\left(\frac{9}{2}\right) + \frac{\pi}{8}
\end{aligned}$$

Example (2)

Find the indefinite integral of $\int \frac{4x+1}{2x^3+x^2}dx$

Solution

$$\int \frac{4x+1}{2x^3+x^2}dx$$

Breaking into partial fractions

$$\frac{4x+1}{2x^3+x^2} = \frac{4x+1}{x^2(2x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x+1} = \frac{Ax(2x+1)+B(2x+1)+Cx^2}{x^2(2x+1)}$$

Equating coefficients

$$Ax(2x+1)+B(2x+1)+Cx^2 = 4x+1$$

$x = -\frac{1}{2} \;\Rightarrow\; \frac{1}{4}C = -1 \;\Rightarrow\; C = -4$

$x = 0 \;\Rightarrow\; B = 1$

$A + 2B = A + 2 = 4 \;\Rightarrow\; A = 2$

$\therefore \frac{4x+1}{2x^3+x^2} = \frac{2}{x} + \frac{1}{x^2} - \frac{4}{2x+1}$

$$\int \frac{4x+1}{2x^3+x^2}dx = \int\left(\frac{2}{x} + \frac{1}{x^2} - \frac{4}{2x+1}\right)dx$$
$$= 2\int\frac{1}{x}dx + \int\frac{1}{x^2}dx - 2\int\frac{2}{2x+1}dx = 2\ln|x| - \frac{1}{x} - 2\ln|2x+1| + c$$

Integrals with a variable limit

Consider an integral of the form

$$\int_0^x f(t)\, dt$$

The function is given as a function of one variable, here $f = f(t)$, and in the limit there is another variable. We are being asked to find the integral from 0 to x of the function $f = f(t)$. The result will be another function that depends on x. The x here represents a variable limit.

Example (4)

Find $\int_2^x \frac{3}{t^2-1}dt$. Evaluate this function when $x = 5$.

Solution

$$\int_2^x \frac{3}{t^2-1}dt = \int_2^x \frac{3}{2(t-1)} - \frac{3}{2(t+1)}dt$$
$$= \left[\frac{3}{2}\ln|t-1|\right]_2^x - \left[\frac{3}{2}\ln|t+1|\right]_2^x$$
$$= \left[\frac{3}{2}\ln\left|\frac{t-1}{t+1}\right|\right]_2^x$$
$$= \frac{3}{2}\left(\ln\left|\frac{x-1}{x+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right)$$
$$= \frac{3}{2}\left(\ln\left|\frac{x-1}{x+1}\right| - \ln\frac{1}{3}\right)$$

Substituting $x = 5$ we get

$$\int_2^5 \frac{3}{t^2-1}dt = \frac{3}{2}\left(\ln\left|\frac{5-1}{5+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right)$$
$$= \frac{3}{2}\left(\ln\frac{4}{6} - \ln\frac{1}{3}\right)$$
$$= \frac{3}{2}\ln 2$$
$$= 1.04 \quad (2 \text{ D.P.})$$

Functions with variable limits frequently occur in the context of physics.

Example (4)

A particle Q is in situated in an electric field. The force acting on this particle is a function of the $r =$ the distance of the particle from the centre of the electric field, and is given by

$$F = \frac{k}{r^2}$$

where k is a constant. The work done on moving the particle from $r = a$ to $r = b$ is

$$U = \int_a^b F\, dr$$

Find the work done in terms of x when a particle is moved from $r = 1$ to $r = x$.

Solution

$$U = \int_1^x F\, dr$$
$$= k\int_1^x r^{-2}\, dr$$
$$= k\left[-r^{-1}\right]_1^x = k\left(-\frac{1}{x} + \left(\frac{1}{1}\right)\right) = k\left(1 - \frac{1}{x}\right) = k\left(\frac{x-1}{x}\right)$$

Example (5)

A force acting on a particle is given by the function

$$f(y) = \frac{y}{1+y}dy$$

The energy of this particle is

$$U(x) = \int_0^x f(y)\,dy$$

Find this energy as a function of x.

Solution

$$\int \frac{y}{1+y}\,dy = \int \frac{1+y-1}{1+y}\,dy$$
$$= \int \left(1 - \frac{1}{1+y}\right) dy$$
$$= \int 1\,dy - \int \frac{1}{1+y}\,dy$$
$$= y - \ln|1+y| + c$$

Hence

$$U(x) - \int_0^x \frac{y}{1+y}\,dy = \left[y - \ln|1+y| + c \right]_0^x$$
$$= x - \ln|1-x|$$