

# Introduction to Linear Programming

## Linear programming in two dependent variables

Linear programming is used to solve, for example, problems in business and industry where a process is subject to constraints and the aim is to maximise some “objective” – such as sales, profit or output.

We will introduce this, firstly, as a purely mathematical problem, and then look at practical applications.

As an exercise in “pure” mathematics, then, a problem in linear programming is presented as follows:

### Objective function

Maximise  $z = 8x + 9y$

### Constraints

Subject to  $x > 0$   
 $y > 0$   
 $2y + x \leq 8$   
 $y + 3x \leq 9$

In the problems we will deal with at this level the objective function will depend on only two variables. In practice, the objective function could depend on many more variables. When there are just two variables, the problem lends itself to graphical solutions by drawing graphs of inequalities, and to algebraic solutions by solving pairs of simultaneous equations. However, when there are more than two variables it ceases to be possible to visualise the relationships graphically, and a systematic method of searching for a solution algebraically has to be introduced. This systematic method is called the Simplex Method, which is the subject of a later unit. However, for the present we work on simpler problems.

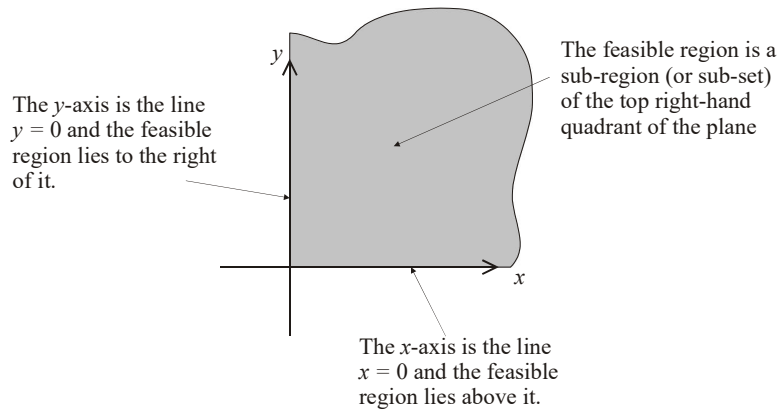
### Solution

The first stage is to graphically display the region where the constraints can be satisfied. This is called the “feasible region”. We begin by simply draw the lines that mark the boundaries of the constraints



$$\begin{aligned}
 x &= 0 \\
 y &= 0 \\
 2y + x &= 8 \\
 y + 3x &= 9
 \end{aligned}$$

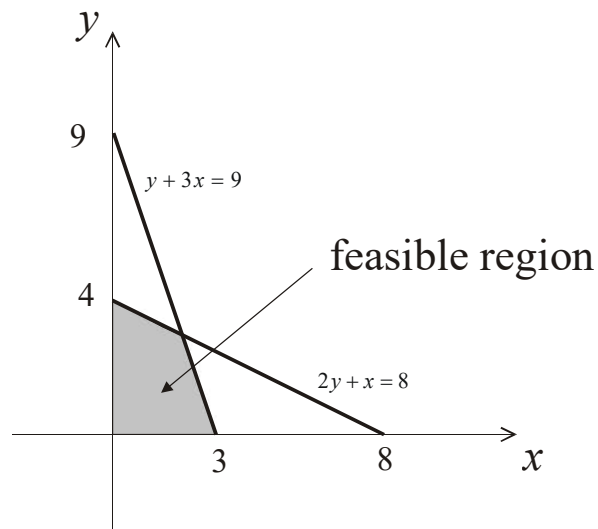
and find the regions where the inequalities are satisfied. The first equations are the equations of the  $x$  and  $y$ -axes respectively, and the feasible region lies above the  $x$ -axis and to the right of the  $y$ -axis respectively.



The lines

$$\begin{aligned}
 2y + x &= 8 \\
 y + 3x &= 9
 \end{aligned}$$

are found in the usual way of plotting two points and joining them, or by rearranging into the form  $y = mx + c$ .



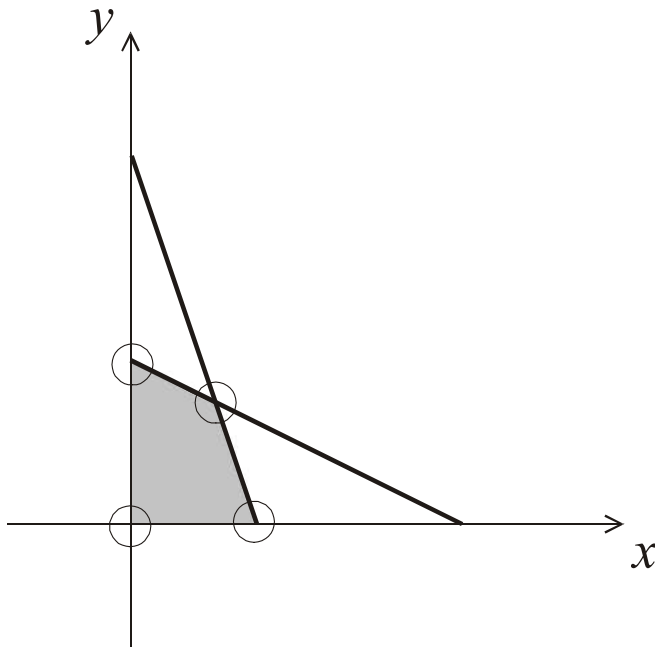
The solution to the problem is just a point lying on the boundary of this feasible region where the value of the objective function



$$z = 8x + 9y$$

is greatest.

Just looking at the graph of the feasible region we can see that there are four possible points. The problem is to determine at which of these four points the objective function is greatest.



We draw a line corresponding to the value of the objective function

$$z = k$$

Where  $k$  is some convenient number. Here, this is the line

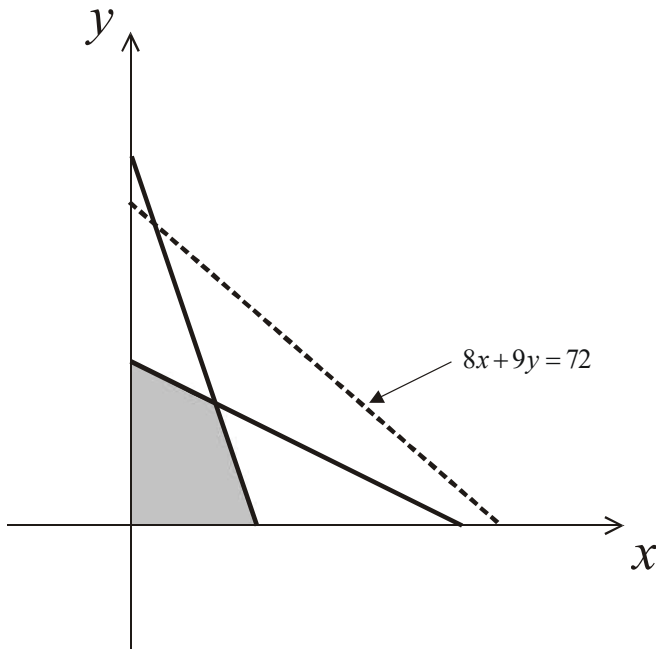
$$8x + 9y = k$$

Since, 8 and 9 are factors of 72, we choose  $k = 72$ , which gives the line

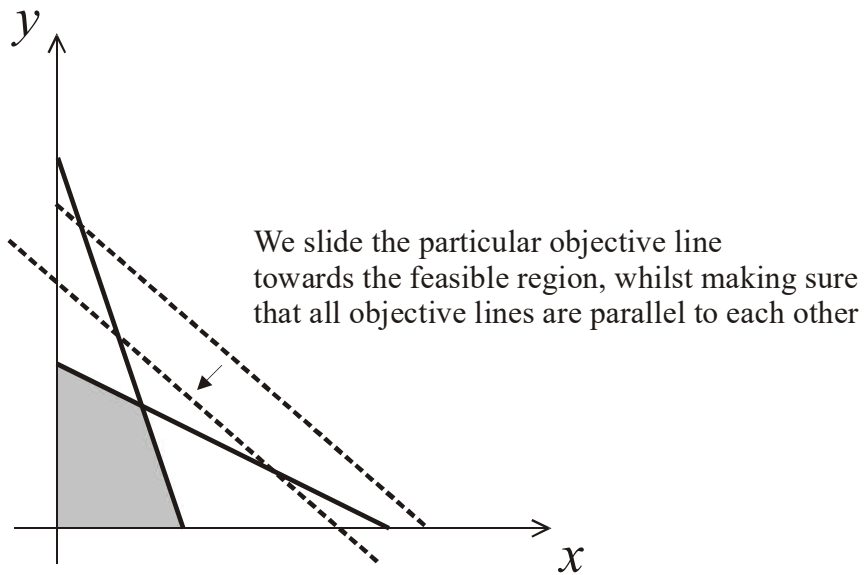
$$8x + 9y = 72$$

which joins the points  $(0,8)$  and  $(9,0)$



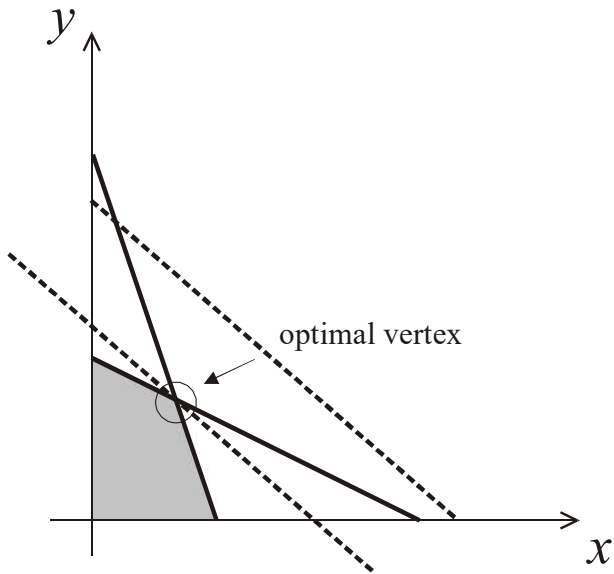


All points on this line lie clearly outside the feasible region. However, we now imagine sliding the line in this case “towards” the feasible region, whilst maintaining the same gradient, until we touch the feasible region at one of its vertices.



The optimal vertex is picked out by this method.



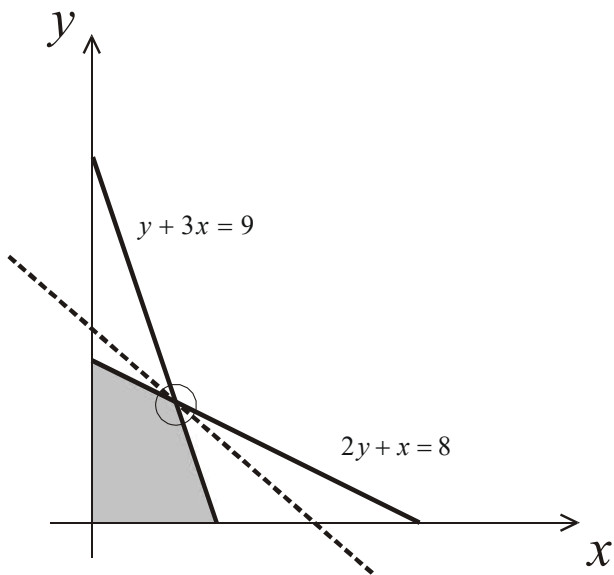


The next stage is to find the coordinates of the optimal vertex and the value of the objective function at those coordinates.

Our graphical method has indicated that the optimal vertex is the solution to the simultaneous equations

$$2y + x = 8$$

$$y + 3x = 9$$



So we solve these simultaneous equations to obtain



$$\begin{aligned}
2y + x &= 8 && (1) \\
y + 3x &= 9 && (2) \\
6y + 3x &= 24 && (1) \times 3 = (3) \\
5y &= 15 && (3) - (2) \\
y &= 3 \\
x &= 2
\end{aligned}$$

At the point (2,3)

$$\begin{aligned}
z &= 8x + 9y \\
&= (8 \times 2) + (9 \times 3) = 51
\end{aligned}$$

And this is our solution to the original problem.

### Formulation of a problem in linear programming

To make effective use of linear programming in business and engineering applications, we need to be able to turn a practical problem into a problem in pure mathematics – that is, we need to translate the practical information into equations.

This process is best shown by illustration.

#### Example

A cocktail can be mixed from a combination of gin and martini. There is a total of 600 *dl* of gin available and 750 *dl* of martini available. Two different cocktails can be prepared per 30*dl* measure of cocktail according to the following chart

	gin/ <i>dl</i>	martini/ <i>dl</i>
martini dry	10	20
gin martini	15	15

The objective is to maximise the total number of measures of cocktails available for sale.

Formulate the problem as a problem in linear programming and solve it.

Solution



Let the number of measures of martini dry be  $x$

Let the number of measures of gin martini be  $y$

The the objective function is

Maximise  $z = x + y$

Using the column in the chart for the amount of gin available, then the constraint that only 600dl of gin is available translates into the inequality

$$10x + 15y \leq 600$$

Likewise, the constraint on the available martini gives the inequality

$$20x + 15y \leq 750$$

We cannot use negative amounts of gin and martini, so we also have the constraints

$$x \geq 0$$

$$y \geq 0$$

So the entire problem can be expressed as

#### Objective function

Maximise  $z = x + y$

#### Constraints

Subject to  $10x + 15y \leq 600$

$$20x + 15y \leq 750$$

$$x \geq 0$$

$$y \geq 0$$

To solve the problem we draw a graph representing the feasible region. Firstly, we determine the lines corresponding to the equations

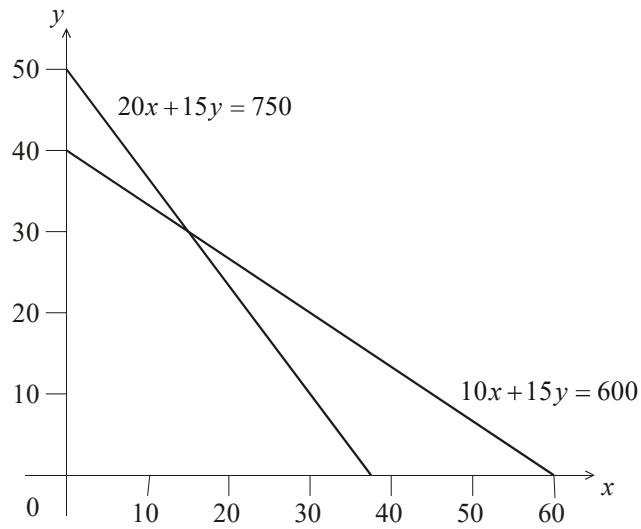
$$10x + 15y = 600$$

$$20x + 15y = 750$$

$$x = 0$$

$$y = 0$$





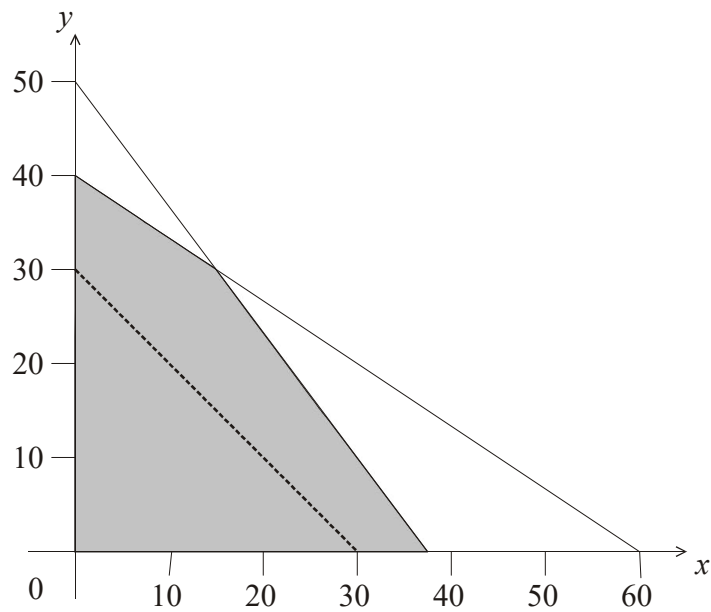
We then shade in the feasible area and find an example of an objective line.  
The objective line is

$$z = x + y$$

and we will plot, for example, the line

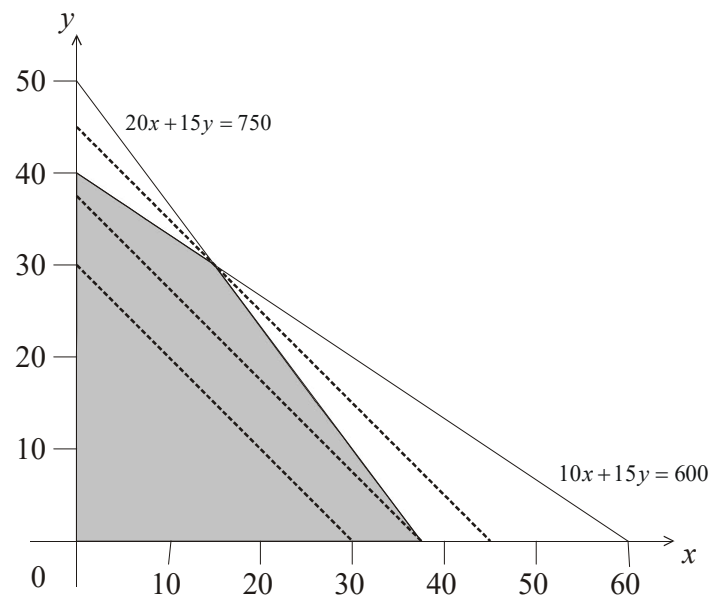
$$x + y = 30$$

which passes through the points  $(0, 30)$  and  $(30, 0)$





We now “slide” the objective line through a sequence of successive lines each parallel to each other to determine the optimal vertex within the feasible region.



So the optimal vertex is the solution to the simultaneous equations

$$10x + 15y = 600 \quad (1)$$

$$20x + 15y = 750 \quad (2)$$

With solution

$$20x + 30y = 1200 \quad (3) = (1) \times 2$$

$$15y = 450 \quad (3) - (2)$$

$$y = 30$$

$$x = 15$$

So we make 15 martini dries and 20 gin martinis.

