Introduction to Complex Numbers

Complex Numbers as the solution to Quadratic Equations with Negative Discriminant

The roots of the quadratic equation

$ax^2 + bx + c = 0$

are given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The term $\Delta = b^2 - 4ac$ is called the discriminant. When $\Delta < 0$ the quadratic equation does not have real roots. Graphically this means that the quadratic function $y = ax^2 + bx + c$ does not cross the x – axis. We now define a number

$i = \sqrt{-1}$

With the aid of this number every quadratic function can be shown to have roots. All terms involving a negative square root can be written in terms of the number $i = \sqrt{-1}$. For instance

$\sqrt{-9} = 3i$

Numbers such as $3i$ that involve the symbol i multiplied by a real number are said to be *imaginary numbers*. Numbers that are made by adding a real number to an imaginary number are said to be *complex numbers*. So a complex number is a number with a real and an imaginary part. We often use the symbol z to denote a complex number, and we write

$z = x + iy$

We illustrate the use of complex numbers to solve quadratic equations:

Example (1)

Find the roots of $x^2 - 4x + 7 = 0$ expressing your answer as complex numbers.

Solution

Substitution into the quadratic formula gives

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16-28}}{2} \\ &= \frac{4 \pm \sqrt{-12}}{2} = \frac{4 \pm 2\sqrt{-3}}{2} \\ &= 2 + \sqrt{3}i \text{ or } 2 - \sqrt{3}i \end{aligned}$$

Philosophical aside (optional)

There may be some question as to why we introduce numbers such as i (the square root of –1), which we cannot visualise. To answer this, let us make two points. Firstly, the introduction of complex numbers enables us to factorise all quadratic functions, and indeed all polynomial functions[1], into linear factors, that is, factors of the form $(x-\alpha)$; thus, there is a sense in which complex numbers are part of the set of all numbers – they complete them. We may think of them as an extra dimension to the world of numbers. Secondly, models in physics of real world phenomena make extensive use of complex numbers – so complex numbers are intensely practical and very useful. Standard examples are phases of currents and voltages in electrical circuits, and they are also used in quantum mechanics.

Addition and Subtraction of Complex Numbers

To add complex numbers, add the components in Cartesian form

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

Example (2)

Let $z_1 = 3 + 4i,\ z_2 = -1 - i$. Find $z_1 + z_2$

Solution

$$z_1 = 3 + 4i,\ z_2 = -1 - i$$
$$z_1 + z_2 = (3 + 4i) + (-1 - i) = (3 - 1) + i(4 - 1) = 2 + 3i$$

Subtraction follows the obvious rule:

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2)] \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

Example (3)

Find $z_1 - z_2$ when $z_1 = 3 + 4i,\ z_2 = -1 - i$

Solution

$$z_1 - z_2 = (3 + 4i) - (-1 - i) = (3 + 1) + i(4 + 1) = 4 + 5i$$

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[1] A polynomial function is a function of the form $f(x) \equiv a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $a_n, a_{n-1}, .., a_1, a_0$ are real numbers. The result referred to here is that all polynomials may be completely factorised provided that the roots are allowed to include complex numbers. The proof is advanced (it is called *the fundamental theorem of algebra*).