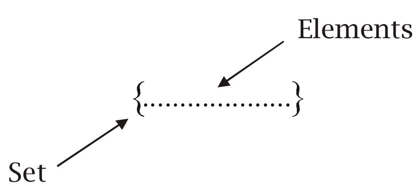
Introduction to Set Theory and Venn Diagrams

What is a set?

A set is a mathematical term for a collection of objects. The objects that comprise a set are called *elements* or *members* of the set. Whenever we think of a collection of objects we form a set. For example, the set of all continents is the collection of landmasses comprising Africa, Asia, Europe, America, Australia and Antarctica. These six continents are the elements or members of the set. To show that they form a set we use curly brackets.

Continents = {Africa, Asia, Europe, America, Australia, Antarctica}

The curly brackets indicate the set and the items inside the curly brackets are the elements of the set.



Example (1)

List all the members of the set of planets

Solution

Planets = {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto}

Defining a set

Sets can be described either by *explicitly listing* all their elements, so that it is the collection of precisely those elements, or by giving a *rule for membership* of the set. For example, we could define the set of continents by listing its elements.

{Africa, Asia, Europe, America, Australia, Antarctica}

or by giving a rule, which would be a definition of a continent.

{Any of the world's main continuous expanses of land}

The second method specifies a *property* that all the members (elements) of the set possess; so all members of a set in such a case must have the property in common. Sometimes, when defining a set by means of a property, we make that property *implicit*. For example

$\{0, 2, 4, 6, 8, 10, ...\}$

defines implicitly the set of all positive even numbers[1]. This approach can be ambiguous, because although we can carry on the list by adding the other even numbers, it is possible to continue such a list in different ways. However, the use of dots to indicate that the list is carried on in some obvious way is quite common.

Example (2)

What is the property implicitly used when defining the set

$\{0, 3, 6, 9, 12, 15, ...\}$

Solution

Positive whole number divisible by three.

Finite and infinite sets

Sets can contain either finite or infinite members. The set of continents is finite – there are exactly six members. The set of all even numbers is infinite – there is no way to finish listing them all. For finite sets any well-defined collection may be regarded as a set. So the collection

{cat, blue, Venus, pine cone, mathematics}

may be regarded as a set even though its members have no obvious connection. So the formation of a *finite* set can be wholly arbitrary, and the common property becomes simply membership of the set. However, when forming an *infinite* set then the definition must specify a common property of the set. The sets that we shall consider here shall mainly be *finite* sets made up of either numbers or objects that we shall list by letters. The expression

$A = \{a, b, c\}$

defines a finite set A by listing its members a, b and c.

Set membership and symbolism for sets

We use the symbol $\in$ to mean that an object is a member of a set.

Africa $\in$ Continents

This is read "Africa is a member of the set of Continents".

$a \in \{a, b, c\}$

[1] We are counting 0 as a positive whole number.

This is read, “*a* is a member of the set comprising the elements *a, b, c*”.

We use $\notin$ to mean that an object is not a member of a set.

$3 \notin \{0,\ 2,\ 4,\ 6,\ 8,\ 10,\ ....\}$

This read, “3 is not an element of the set of all positive even numbers”.

When we specify a set by means of a property we use the following notation

$\text{Continents} = \{x : x \text{ is one of the world's main continuous expanses of land}\}$

The colon inside the curly brackets is read “such that” and introduces the property that defines membership of the set. This is read, “The set Continents is set of all object *x* such that *x* is one of the world’s main continuous expanses of land”. Sometimes, a vertical stroke is used instead of a colon – this is an alternative notation.

$A = \{x | x \text{ is an even number}\}$

The vertical stroke has exactly the same meaning as the colon, and is also read, “such that”.

Example (3)

(*a*) Translate the following using the symbol $\in$.

Saturn is a planet

(*b*) Translate the following into words

$5 \notin \{x : x \text{ is even}\}$

Solution

(*a*) $\text{Saturn} \in \text{Planets}$

or

$\text{Saturn} \in \{\text{Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto}\}$

(*b*) 5 is not a element of the set of all even numbers

or, more simply

5 is not an even number

Equality of sets

Two sets are *equal* (or *identical*) if they have exactly the same elements regardless of the order in which they are listed. Sets are collections of objects without regard to the order in which those objects occur. Thus, $\{\text{red, white, blue}\} = \{\text{white, blue, red}\}$. These are the same set.

Example (4)

One of the following statements is true and one is false. Determine which is which.

(1) $\{a,b,c\} = \{b,c,a\}$

(2) $\{a,b,c\} \neq \{c,a,b\}$

Solution

(1) is true, and (2) is false.

The null or empty set

We can also form a set with no objects. This is called the *null* or *empty set*. There is only one null set, because there is only one set with nothing in it! It is denoted by the symbol $\varnothing$.

Domain of discourse

Every definition of a set implies a *domain of discourse*. This is the largest collection of objects to which the members of the set can belong. Usually it will be clear from the context what the domain of discourse is. The domain of discourse is represented by the symbol V.

Sets of numbers

When the objects in the set are numbers, it is usual to specify the type of number that is allowed and this becomes the domain of discourse. We also have special symbols to denote these types of numbers.

$\mathbb{N} = \{\text{the set of all natural numbers}\} = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ .....\}$

$\mathbb{Z} = \{\text{the set of all positive and negative integers}\} = \{....-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ ....\}$

$\mathbb{Q} = \{\text{the set of all rational numbers}\}$

$= \left\{\text{the set of all numbers of the form } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers}\right\}$

$= \left\{x : x = \frac{p}{q}, \text{ where } p,\ q \in \mathbb{Z}\right\}$

$\mathbb{R} = \{\text{the set of all real numbers}\}$

Rational numbers include all fractions, but there are numbers that cannot be written as fractions, such as √2. Such numbers have infinite, non-repeating decimal expansions. The real numbers include these. For example, we could define the set of positive even numbers as follows

$A = \{x \in \mathbb{N} : x \text{ is even}\}$

This is read, "*A* is the set of all natural numbers *x* such that *x* is even".

Order of a set

The number of elements in a set is called the *order* of the set, and is denoted by $n(A)$.

When $A = \{a, b, c\}$, then $n(A) = 3$.

When a set is infinite its order is denoted by ∞.

When $A = \{x \in \mathbb{N} : x \text{ is even}\}$, then $n(A) = \infty$.

Example (5)

(a) The set Planets is defined by

$$\text{Planets} = \{\text{Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto}\}.$$

Find $n(\text{Planets})$.

(b) The set X is defined by

$$A = \{x \in \mathbb{N} : x \text{ is divisible by } 5\}.$$

Find $n(X)$.

Solution

(a) $n(\text{Planets}) = 9$

(b) $n(X) = \infty$

Subsets

If B is a subset of the set A then all the elements of B are also elements of A. The symbol for subset is $\subseteq$. Thus

$$B \subseteq A$$

is read, "B is a subset of A". The definition of the subset property is formally expressed by

$$B \subseteq A \Leftrightarrow \text{if } x \in B \text{ then } x \in A$$

Example (5)

List all the subsets of the set

$$A = \{a, b, c\}$$

How many subsets are there?

Solution

$A = \{a, b, c\}$

Subsets of A $\{a, b, c\}$

$\{a, b\}$

$\{a, c\}$

$\{b, c\}$

$\{a\}$

$\{b\}$

$\{c\}$

$\varnothing$

There are $2^3 = 8$ subsets.

There are two things about the solution to example 5 that may surprise you.

(1) That the set $A = \{a, b, c\}$ is a subset of itself.

$A \subseteq A$

This is true of *any* set. Every set is a subset of itself. This follows strictly from the definition of what a subset is for $A \subseteq A$ just says every element of A is an element of A.

(2) That the null set, $\varnothing$, is also a subset of A. This is also true for any set. The null set is a subset of every set whatsoever. Once again, this follows from the definition of what a subset is. The null set contains *no elements* and so every element of the null set (that is no element) is also an element of the set A.

Venn diagrams

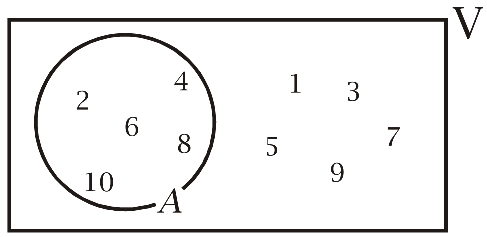
Venn diagrams are diagrammatic representations of the relationship between sets and subsets. A rectangle generally represents the domain of discourse Subsets are generally represented by circles or other closed figures, such as squares, rectangles, ellipses and so forth. Elements in a set are placed inside the circle. Elements not in the set are placed outside the circle. It is a convenient diagrammatic way of representing sets and their members.

Example (6)

Let $\mathrm{V} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{x \in \mathrm{V} : x \text{ is even}\} = \{2, 4, 6, 8, 10\}$

(*a*) Draw a Venn diagram representing this information.

Solution



Remark

The symbol V here stands for the domain of discourse, and contains all the objects (here, numbers) listed, including those in the subset *A*. Any space between the numbers has no significance. No relations are indicated by the relative positions of the objects. A set is represented simply by placing a container (a rectangle, circle or other object) around a collection of objects.

Complement of a set

The *complement* of *A* is the set of elements of V (the universe of discourse, or domain) that are not elements of *A*. The complement of *A* is denoted by the symbol A'.

Example (6) continued

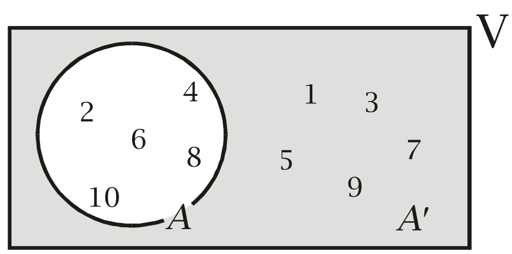
Given $\mathrm{V} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{x \in \mathrm{V} : x \text{ is even}\} = \{2, 4, 6, 8, 10\}$

(*b*) Find A' and mark A' onto the Venn diagram drawn in the solution to part (*a*)

Solution

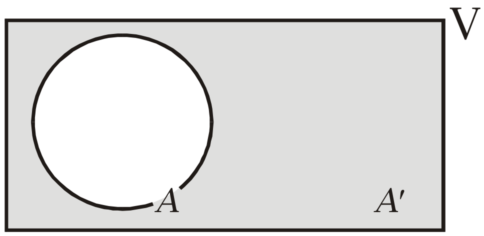
$A' = \{1, 3, 5, 7, 9\}$



A formal definition of the complement is

$A' = \{x \in \mathrm{V} : x \notin A\}$

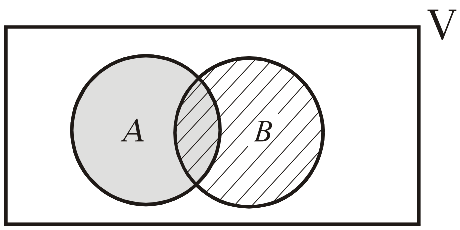
This is read, "The complement of *A* is the set of all elements of the universe of discourse, V, such that none of these elements are elements of *A*". The Venn diagram representation of a complement of a set *A* is



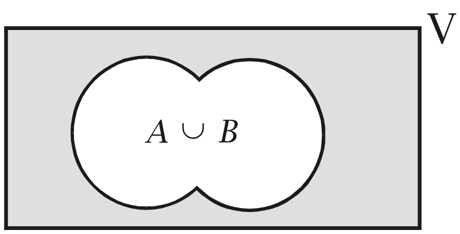
The shaded area represents the complement of A.

Union and intersection

The union of two sets, A and B is the set of elements that are elements of *either* A or B or both. Its symbol is $A \cup B$. In a Venn diagram if the two separate sets are shown as follows



then the union of A and B is shown by

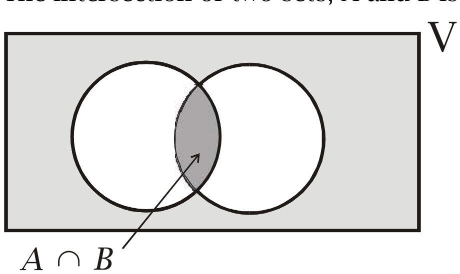


Formally, the definition of the union is

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

Intersection of two sets

The intersection of two sets, A and B is the set of elements that are elements of both A and B.



The formal definition is

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

Example (7)

$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 2, 5, 7, 8, 9, 10\}$

$B = \{2, 6, 7\}$

Find

(a) A',

(b) $A \cup B$

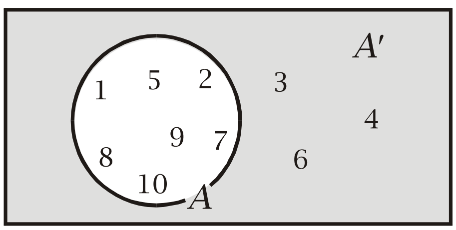
(c) $A \cap B$

(d) $A' \cap B$

and show these sets using Venn diagrams.

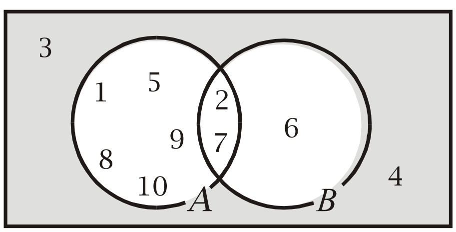
Solution

(a) $A' = \{3, 4, 6\}$

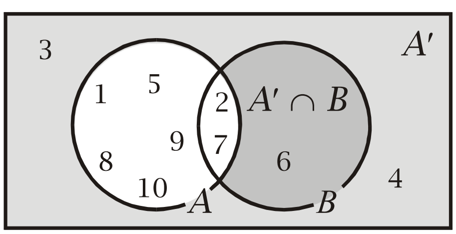


(b) & (c) $A \cup B = \{1, 2, 5, 6, 7, 8, 9, 10\}$

$A \cap B = \{2, 7\}$



(d) $A' \cap B = \{6\}$



Example (8)

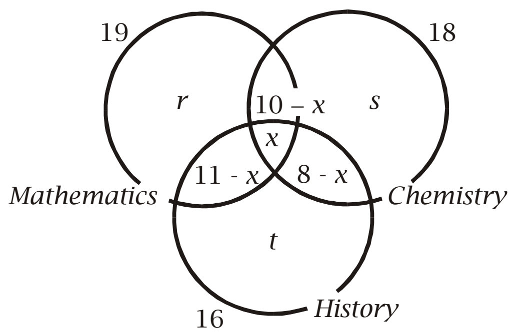
A class has 30 students, 16 of which take history, 19 take mathematics, 18 take chemistry. 10 students take both mathematics and chemistry and 11 students take mathematics and history, and 8 students take history and chemistry.

(*i*) How many students took all three subjects?

(*ii*) How many students took each subject?

Solution

We begin by drawing a Venn diagram representing the information contained in the question.



Then from the diagram

Let x be the number of students taking all three subjects.

Let s, r and t be the number of students taking only chemistry, only mathematics and only history respectively. Then from the diagram we have.

$r + 10 - x + x + 11 - x = 19 \quad \Rightarrow \quad r = x - 2$

$s + 10 - x + x + 9 - x = 18 \quad \Rightarrow \quad x = s$

$t + 11 - x + x + 8 - x \quad \Rightarrow \quad t = x - 3$

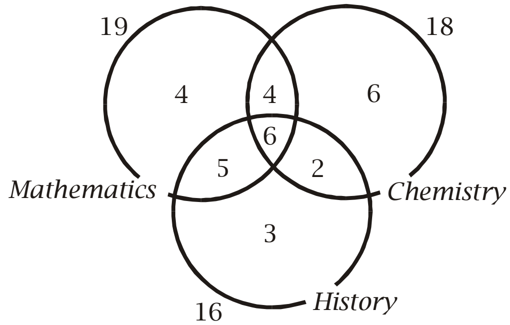
$r + s + t + 10 - x + 11 - x + 8 - x + x = 30 \quad \Rightarrow \quad r + s + t - 2x = 1$

The solution to these equations is

$x - 2 + x + x - 3 - 2x = 1$

$x = 6$

$s = 6, \quad r = 4, \quad t = 3$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18
4
4
6
6
5
2
Mathematics
Chemistry
3
16
History