Introduction to the Differential Calculus

Prerequisites

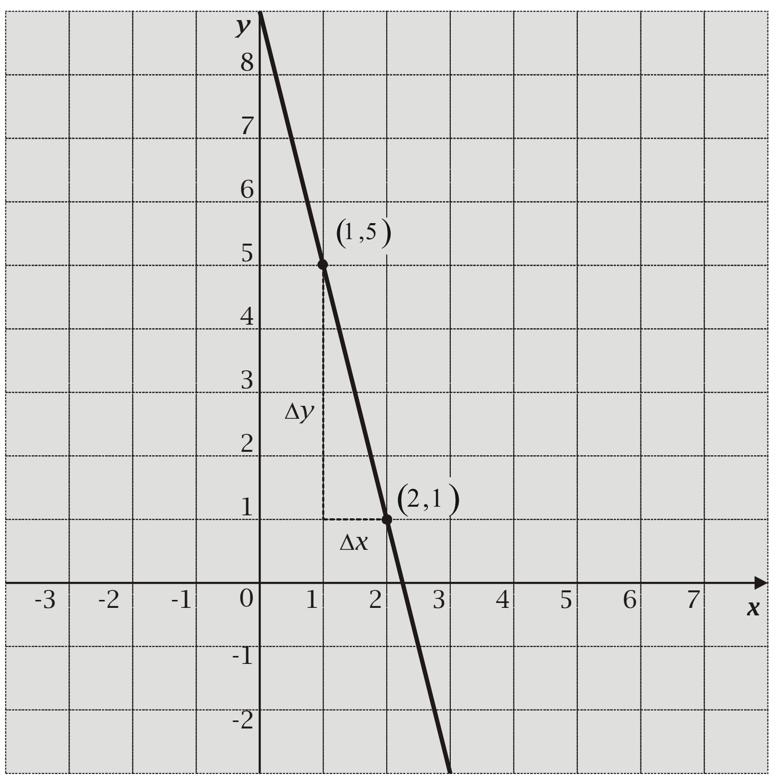
You should be familiar with the Cartesian equation of the straight-line. Let us revise this.

Example (1)

In the following find the values of m and c if the straight line $y = mx + c$ passes through the points $(1,5)$ and $(2,1)$.

Solution

Let us first sketch the graph



$$m = \frac{\Delta y}{\Delta x} = \frac{(5-1)}{(1-2)} = -\frac{4}{1} = -4$$

On substituting $m = -4$ into

$y = mx + c$

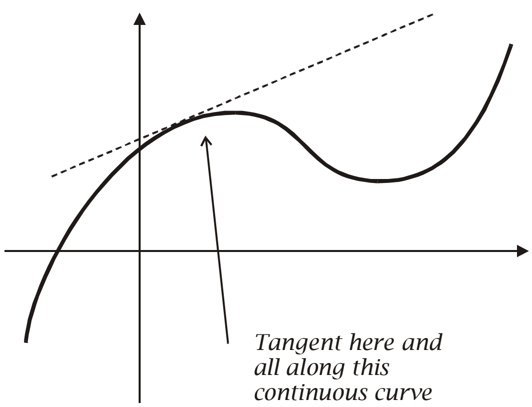
we get
$y = -4x + c$
This goes through the point $(2,1)$
On substituting $x = 2,\ y = 1$
$1 = -4 \times 2 + c$
$c = 9$
Hence
$y = -4x + 9$

Please pay particular attention in this solution the expression for the gradient m that uses the symbol $\frac{\Delta y}{\Delta x}$ where

$$\text{gradient} = m = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}.$$

Gradients of curves

The gradient of a straight-line is constant   Smooth curves also have gradients. A tangent is a line just touching a curve.



We can use graphical methods to approximate the gradient of a tangent to a curve at a given point, as the following example illustrates.

Example (2)

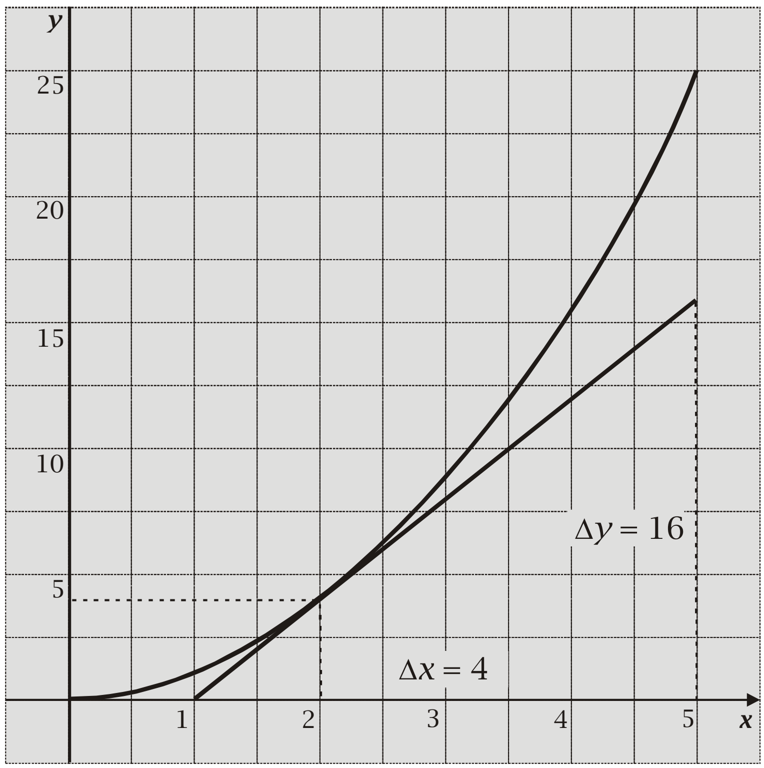
Plot the graph of $y = x^2$ between $x = 0$ and $x = 5$. Draw a tangent at $x = 2$ and use a graphical method to find the approximate gradient of this tangent. Comment on the level of accuracy obtained by this method.

Solution

First we need to tabulate values of $y = x^2$ between $x = 0$ and $x = 5$.

x	**0**	**1**	**2**	**3**	**4**	**5**
$y = x^2$	0	1	4	9	16	25

The graph together with the tangent at $x = 2$ follows.



From the graph the gradient of the tangent at $x = 2$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{16}{4} = 4$$

The question about level of accuracy obtained by this method is important. Clearly, the accuracy of this method is limited. We could say that we have found the gradient at $x = 2$ within a margin of error of ± 0.25. However, this is not a precise estimate of the margin of error either, but a guess at it.

The example demonstrates the obvious limitation of this graphical method. The *differential calculus* is a theory for finding the gradients to curves that overcomes the limitation of this method.

The derivative

In the previous example we took the function $y = x^2$ and found its gradient by a graphical method at $x = 2$. We could repeat this method for the function $y = x^2$ at every point *x*.

Example (3)

A table of the values of the function $y = x^2$ and the *exact* value of the gradient of its tangents is given as follows.

x	**0**	**1**	**2**	**3**	**4**	**5**
$y = x^2$	0	1	4	9	16	25
gradient function	0	2	4	6	8	10

Try to *conjecture* what the function representing this gradient is.

Solution

The *gradient function* is going up in steps of 2 for each increase of 1 unit in *x*. Therefore, we conjecture that the gradient function is

$y = 2x$

This conjecture is true. Starting with the function

$y = f(x) = x^2$

we can *derive* the function that gives its gradient at any given point; this is

$f'(x) = 2x$

The symbol $f'(x)$ denotes this gradient function, which is also called the *derivative*. To *differentiate* a function $y = f(x)$ means to obtain its *derivative* – the function $f'(x)$.

Example (4)

You are given that the derivative of the function $y = f(x) = x^2$ is $f'(x) = 2x$.

Find the value of the gradient of $y = x^2$ at the points $(0.5, 0.25)$ and $(-3.5, 12.25)$.

Solution

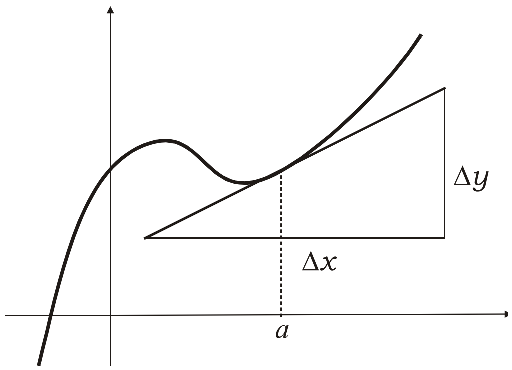
$f'(x) = 2x$

$f'(0.5) = 1$

$f'(-3.5) = -7$

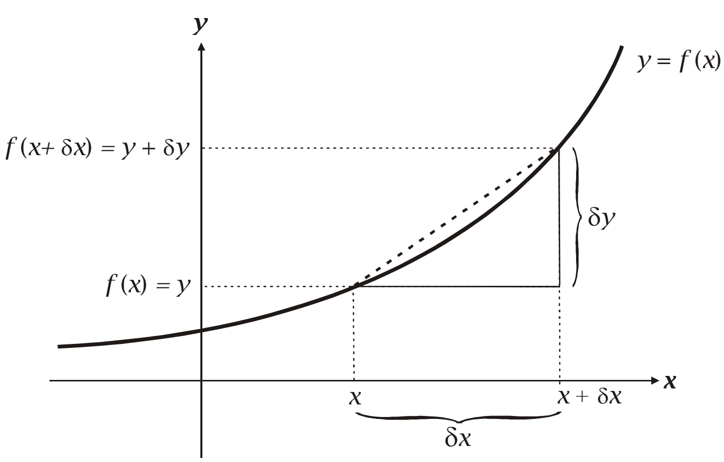
Differential Calculus

The gradient of the tangent at a of a function $f(x)$ represents the instantaneous rate of change of that function at *a*.



$$\text{gradient} = m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x}$$

The differential calculus provides a mathematical means of arriving at the gradient of a tangent to a function. Firstly, we find an expression for the gradient of a *cord* joining two points on the graph of the function: To do so we start with the following diagram for a function $y = f(x)$.



It is important to study this diagram carefully. In the diagram the symbols δx and δy stand for small increases in the values of x and y respectively along a cord. The Greek letter δ ("delta") is used for this purpose. It is similar to the use of the Greek capital letter Δ to stand for "change in". The smaller letter is used to capture the idea that we are looking at a *small* change in x and the corresponding small change in y.

Example (5)

Let $y = f(x) = x^2$ and let $x = 2$ and $\delta x = 0.1$.

Find $x + \delta x, f(x + \delta x)$ and δy.

Solution

$x + \delta x = 2 + 0.1 = 2.1$

$f(x + \delta x) = f(2.1) = (2.1)^2 = 4.41$

$\delta y = f(x + \delta x) - f(x) = 4.41 - 4 = 0.41$

Now take another look at the previous diagram. The gradient of the cord joining (x, y) to $(x + \delta x, y + \delta y)$ is

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Example (5) continued

With the same arguments as in the previous example, find the gradient of the cord joining (x, y) to $(x + \delta x, y + \delta y)$.

Solution

This is

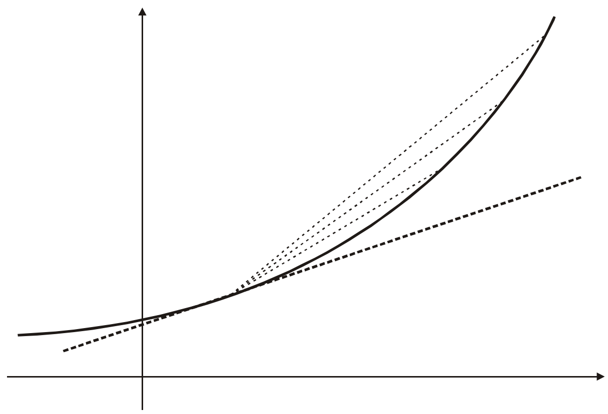
$$\begin{aligned} \frac{\delta y}{\delta x} &= \frac{f(x + \delta x) - f(x)}{\delta x} \\ &= \frac{4.41 - 4}{0.1} \\ &= 4.1 \end{aligned}$$

Note: Sometimes the symbol h (or some other letter) is used for the small increment; in this case the gradient is given by

$$\text{gradient} = \frac{\delta y}{\delta x} = \frac{f(x + h) - f(x)}{h}$$

It is the same idea, just with a different symbol. The use of h has the benefit of emphasising that $\delta x = h$ stands for a single quantity. In the symbol δx it is **not** possible to split δ from x. The δx symbol has the advantage of better representing the idea of a small change in the x value.

The differential calculus arises from the following idea. As δx gets smaller and smaller, the cord joining (x, y) to $(x+\delta x, y+\delta y)$ gets closer and closer to the tangent at *x*. The following diagram illustrates this idea. It shows the length of a cord at a certain point getting shorter and shorter and so *converging* on the tangent at that point.



Example (5) continued

In the previous question we found the gradient of the cord joining $(2,4)$ to $(x+\delta x, y+\delta y)$ where $\delta x = 0.1$ to be

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x} = 4.1$$

(*a*) Find the gradient of the cord joining $(2,4)$ to $(x+\delta x, y+\delta y)$ where

(i) $\delta x = 0.01$

(ii) $\delta x = 0.001$

(iii) $\delta x = 0.0001$

(*b*) To what value does the gradient of the cord seem to be approaching?

Solution

(a) (i) $\delta x = 0.01$

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{2.01^2 - 2^2}{0.01} = 4.01$$

(ii) $\delta x = 0.001$

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{2.001^2 - 2^2}{0.001} = 4.001$$

(iii) $\delta x = 0.0001$

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{2.0001^2 - 2^2}{0.0001} = 4.0001$$

(b) We have a sequence of numbers

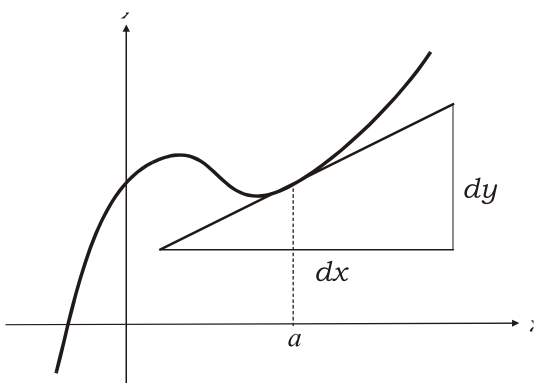
$4.1 \to 4.01 \to 4.001 \to 4.0001$

as $\delta x = 0.1, 0.01, 0.001, 0.0001$

It seems likely that this sequence will converge on the number 4.

This conjecture that as δx gets smaller and smaller the gradient of the cord to $y = f(x) = x^2$ at $x = 2$ gets closer and closer to 4 is correct, so the gradient of the tangent at $x = 2$ is 4.

In general, as $\delta x \to 0$ (as δx gets closer and closer to 0) then the gradient of the cord gets closer and closer to the gradient of the tangent. This is really a statement about the *limit* of the *sequence* of gradients of cords that converge on the tangent. We say that "in the limit as $\delta x \to 0$ (as δx tends to 0) the gradient of the cord $\to$ (tends to) the gradient of the tangent." The following diagram shows a tangent to a function $y = f(x)$.



In the diagram we denote the gradient of the tangent by $\frac{dy}{dx}$. Here *dy* and *dx* in $\frac{dy}{dx}$ mean the increments in *y* and *x* respectively along the tangent at a given point *a* to the graph of $y = f(x)$. It is a ratio, and as such the symbols *dy* and *dx* cannot be split. Note the difference of meaning between *dy* and *dx* on the one hand and δx and δy on the other. The symbols δx and δy stand for increments along a *cord*, whereas *dy* and *dx* stand for increments along a *tangent.* The fundamental relationship between these that we have just established is

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left\{ \frac{\delta y}{\delta x} \right\}$$

This expression is the formal way of writing "in the limit as $\delta x \to 0$ the gradient of the cord $\to$ the gradient of the tangent." Now note, the gradient of the tangent, which is $\frac{dy}{dx}$, is what we called earlier the derivative of the function $y = f(x)$, which is $f'(x)$. So what we have established is $f'(x) = \frac{dy}{dx}$.

This says, "the derivative of $y = f(x)$ is the gradient of the tangent." However, we have also shown that

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

so we get the fundamental result that

$$f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0}\left\{\frac{\delta y}{\delta x}\right\} = \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x) - f(x)}{\delta x}\right\}.$$

We call this the formula for *differentiation from first principles.* Note, if we used the symbol h instead of δx the formula would be

$$\frac{dy}{dx} = \lim_{h \to 0}\left\{\frac{\delta y}{h}\right\} = \lim_{h \to 0}\left\{\frac{f(x+h) - f(x)}{h}\right\}$$

δx and h are interchangeable. When you are asked (in this context) to find a derivative from first principles, it is with either form of this equation (or equivalent) that you start.

Example (6)

Find the derivative of $y = f(x) = x^2$ from first principles.

Solution

We give the solution in both forms. They are equivalent.

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x) - f(x)}{\delta x}\right\}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^2)}{dx}\\ &= \lim_{\delta x \to 0}\left\{\frac{(x+\delta x)^2 - x^2}{\delta x}\right\}\\ &= \lim_{\delta x \to 0}\left\{\frac{x^2 + 2x(\delta x) + (\delta x)^2 - x^2}{\delta x}\right\}\\ &= \lim_{\delta x \to 0}\left\{\frac{2x(\delta x) + (\delta x)^2}{\delta x}\right\}\\ &= \lim_{\delta x \to 0}\{2x + \delta x\}\\ &= 2x\end{aligned}$$

$$\frac{dy}{dx} = \lim_{h \to 0}\left\{\frac{f(x+h) - f(x)}{h}\right\}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^2)}{dx}\\ &= \lim_{h \to 0}\left\{\frac{(x+h)^2 - x^2}{h}\right\}\\ &= \lim_{h \to 0}\left\{\frac{x^2 + 2xh + h^2 - x^2}{h}\right\}\\ &= \lim_{h \to 0}\left\{\frac{2xh + h^2}{h}\right\}\\ &= \lim_{h \to 0}\{2x + h\}\\ &= 2x\end{aligned}$$

Let us annotate this solution for the method using δx. The line

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x)-f(x)}{\delta x}\right\}$$

states the principle that we are using to solve the question. In the line

$$\frac{dy}{dx} = \frac{d(x^2)}{dx}$$

the function $f(x) = x^2$ is substituted for y. In the next line we apply the principle quoted first to the function $y = f(x) = x^2$ so we are required to substitute $(x+\delta x)^2$ and x^2 for $f(x+\delta x)$ and $f(x)$ respectively to obtain

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left\{\frac{(x+\delta x)^2 - x^2}{\delta x}\right\}$$

We have replaced the general function f by the specific function $f(x) = x^2$. Now there follows some algebra in which the term $(x+\delta x)^2$ is expanded. This makes it possible to cancel out some terms. Then δx is a factor of both the numerator (top) and denominator (bottom) of the fraction, so it is cancelled through. At this point we have

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\{2x + \delta x\}$$

Here we are allowed by the assumption that taking the limit is equivalent to putting $\delta x = 0$ to obtain $\lim_{\delta x \to 0}\{2x + \delta x\} = 2x$. When we "*take the limit*" we substitute $\delta x = 0$ and scratch out the $\lim_{\delta x \to 0}$ part. Note, this step is only allowed if there is no zero in the denominator (the bottom) of a fraction, for otherwise it is equivalent to dividing by zero, and that is not allowed – it leads to contradictions.

Rules for Differentiation

Rules for differentiation provide short-cut methods for finding derivatives.

1. The functions

$$y = f(x) = x^2$$
$$y = f(x) = x^3$$
$$y = f(x) = x^4 \qquad \ldots$$

comprise a family of functions the general expression for which is

$y = x^n$.

The derivative for all of the members of this family follows a single rule

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

or alternatively

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Example (7)

Find the derivative of $f(x) = x^6$.

Solution

$f'(x) = 6x^5$

Remark. To use the definition $f'(x) = nx^{n-1}$, drop the index n down in front of the x^n and lower the index by 1.

2. Functions like $y = f(x) = \frac{1}{x}$ can be written in index form as $y = x^{-1}$. The general form of such functions is

$$y = f(x) = \frac{1}{x^n} = x^{-n}$$

To find these functions, you simply apply the same rule as in (1), that is

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

So the derivative of $\frac{1}{x}$ is

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}.$$

Similarly, functions like $y = f(x) = \sqrt{x}$ can be written in index form as $y = f(x) = x^{\frac{1}{2}}$. These also obey the rule If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ where n is a rational number.

Example (8)

(*a*) Find the derivative of $y = \frac{1}{x^3}$.

(*b*) Find the derivative of $y = x^{-\frac{1}{3}}$, writing your solution in surd form.

Solution

(*a*) In index form this is $y = f(x) = x^{-3}$. The derivative is

$$\frac{dy}{dx} = f'(x) = -3x^{-4} = -\frac{3}{x^4}.$$

(*b*) $y = f(x) = x^{-\frac{1}{3}}$

$$\frac{dy}{dx} = f'(x) = -\frac{1}{3}x^{-\frac{4}{3}} = -\frac{1}{3\sqrt[3]{x^4}}$$

3. We have dealt with functions of the type $y = f(x) = x^n$ where *n* is an integer. Now we ask, what happens if such a function is also multiplied by a real number? For example, what is the derivative of $y = 3x^4$? We know from (1) that the derivative of $f(x) = x^4$ is $f'(x) = 4x^3$. How does the multiplication of this function by 3 affect the outcome? The answer is provided by the *rule for constant multiples*, which is

$$(af(x))' = af'(x)$$

or alternatively

$$\frac{d}{dx}(ay) = a\frac{dy}{dx}.$$

What this says is that if a function $y = f(x)$ is multiplied by a constant multiple *a*, then the derivative is also multiplied by that number.

Example (9)

Find the derivative of $y = f(x) = 3x^4$.

Solution

$f'(x) = 3 \times 4x^3 = 12x^3$

4. Using the rules expressed in (1), (2) and (3) above we can find the derivative of individual expressions such as

$y = x^6$ $\quad\quad y = \frac{1}{x^3}$ $\quad\quad y = 3x^4$

The next question to ask is what happens if such functions are added and subtracted? For example, what is the derivative of $y = x^6 + \frac{1}{x^3} - 3x^4$? The answer to this question is provided by the *rule for sums* of functions, which is

$$(f(x)+g(x))' = f'(x)+g'(x)$$

or alternatively

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x)+\frac{d}{dx}g(x).$$

What this says is that the derivative of the sum of two functions is found by adding together the derivatives of the two functions separately. So if the function is $y = f(x) = x^6 + \frac{1}{x^3} - 3x^4$ then its derivative is equal to the derivative of x^6 added to the derivative of $\frac{1}{x^3}$ added to the derivative of $-3x^4$. That is

$$\frac{dy}{dx} = 6x^5 - 3x^{-4} - 12x^3 = 6x^5 - \frac{3}{x^4} - 12x^3$$

Example (10)

Given $f(x) = g(x) + h(x)$
where $g(x) = x^5$ and $h(x) = 3x^3$
Find $f'(x)$.

Solution

$f(x) = g(x) + h(x) = x^5 + 3x^3$
$g'(x) = 5x^4$
$h'(x) = 9x^2$
$f'(x) = g'(x) + h'(x) = 5x^4 + 9x^2$

Gradients

We have seen that the derivative of a function at a point $x = a$ finds the gradient of that function at that point. The gradient of the function $f(x)$, which is the same as its *rate of change*, at the point $x = a$ is found by evaluating the derivative at $x = a$. This may be written

$$f'(x)\big|_{x=a} = f'(a)$$

or

$$\left.\frac{dy}{dx}\right|_{x=a}$$

Here the vertical line means “evaluated at”. This means to substitute the value $x = a$ into the expression for the derivative. For example, the derivative of the function $y = f(x) = 9 - x^2$ evaluated at $x = 3$ is

$f'(x)\big|_{x=3} = -2x\big|_{x=3} = -6$.

Since we can find the gradient to a function at a given point, we can also find the equation for the tangent to that function at that point.

Example (11)

Find the equation of the tangent to the curve $y = 9 - x^2$ at $x = 2$.

Solution

$y = 9 - x^2$

When $x = 2$ we have $y = 9 - 2^2 = 5$. The derivative is

$$\frac{dy}{dx} = -2x$$

When $x = 2$ $\frac{dy}{dx} = -2 \times 2 = -4$. So this is the gradient of the tangent at $x = 2$. The tangent at $(2,5)$ is a straight-line whose general equation is $y = mx + c$. Here $m = -4$, so we have $y = -4x + c$. To find the value of the intercept *c*, we substitute $x = 2,\ y = 5$ to obtain

$$5 = -4 \times 2 + c$$
$$c = 13$$

So the equation of the tangent at $(2,5)$ to $y = 9 - x^2$ is

$$y = -4x + 13.$$

The term *normal* means perpendicular to the tangent.

Example (12)

Find the equation of the normal to the curve $y = 9 - x^2$ at the point where the curve crosses the positive *x*-axis. Find also the coordinates of the point where the normal meets the curve again.

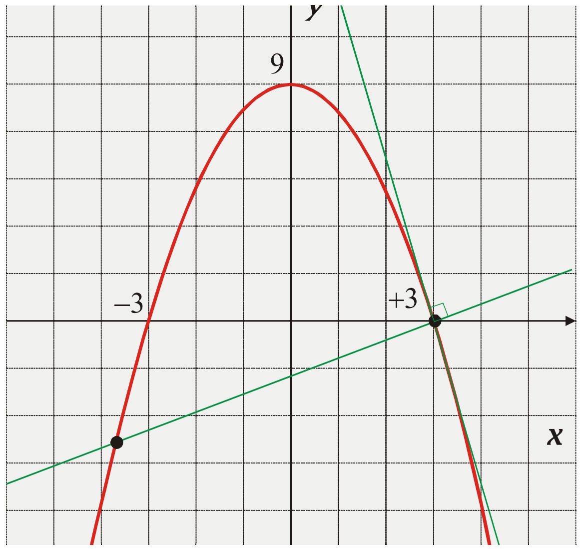
Solution

Firstly, we need to find the point where the curve meets the positive *x*-axis.

$$y = 9 - x^2 = 0$$

$$(3 - x)(3 + x) = 0$$

So the curve meets the positive *x*-axis at $x = 3$. We can sketch the curve.



The tangent just touches the curve at 3 and the normal is perpendicular to it. So we need to find the gradient of the tangent at $x = 3$.

$$\left.\frac{dy}{dx}\right|_{x=3} = \left.\frac{d}{dx}\left(9 - x^2\right)\right|_{x=3} = -2x\big|_{x=3} = -6$$

From here on, we are just solving a problem involving the equation of the straight line and the intersection of lines and curves. The normal is perpendicular to the tangent so the gradient of the normal is given by the rule

$$m_1 m_2 = -1$$

Here

$$m_1 = -6$$

so

$$m_2 = \frac{1}{6}$$

Let the equation of the normal be

$$y = m_2 x + c$$

Hence

$$y = \frac{1}{6}x + c$$

On substituting $x = 3,\ y = 0$ we get

$$0 = \frac{1}{2} + c$$

$$c = -\frac{1}{2}$$

So the equation of the normal is

$y = \frac{1}{6}x - \frac{1}{2}$

We were also asked to find the coordinates of the point where this normal meets the curve again. That is, to solve simultaneously the equations.

$y = \frac{1}{6}x - \frac{1}{2}$

$y = 9 - x^2$

To solve this we can equate y to get

$9 - x^2 = \frac{1}{6}x - \frac{1}{2}$

$54 - 6x^2 = x - 3$

$6x^2 + x - 57 = 0$

To finish off we need the quadratic formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence

$$x_{1,2} = \frac{-1 \pm \sqrt{1 + 1368}}{12}$$

$$= 3 \text{ or } -\frac{19}{6}$$

When $x = -\frac{19}{6}$

$$y = 9 - \left(-\frac{19}{6}\right)^2 = 9 - \frac{361}{36} = -\frac{37}{36}$$

The point of intersection is $\left(-\frac{19}{6}, -\frac{37}{36}\right)$