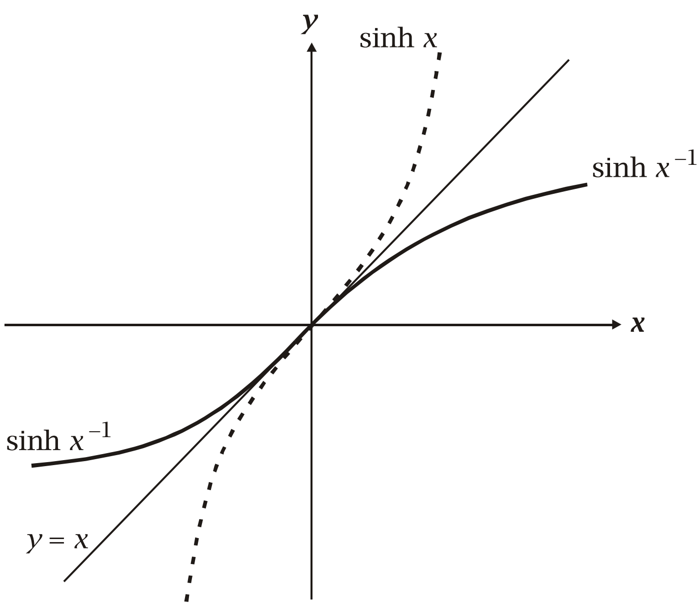
Inverse Hyperbolic Functions

Inverse hyperbolic functions

In this chapter we will extend our knowledge of hyperbolic functions to include inverse hyperbolic functions. To have an inverse a function must be one-one.

sinh *x*

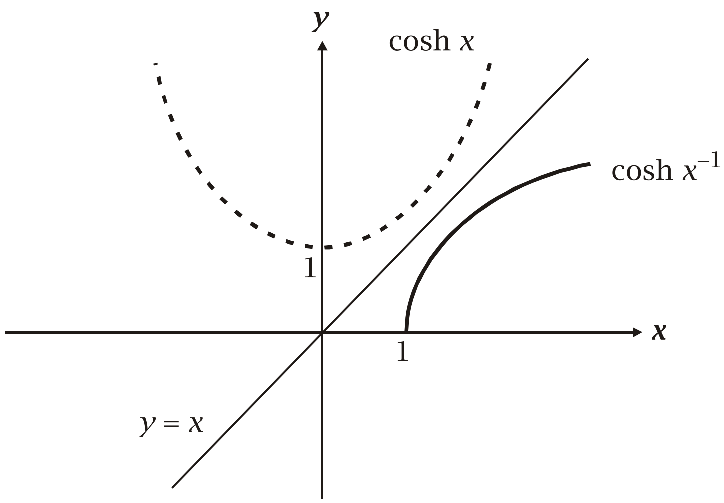
$\sinh x$ is a one-one function and consequently has an inverse, $\sinh^{-1} x$, (also denoted as arcsinh *x*) defined on the whole of $\mathbb{R}$ (its domain is the whole of $\mathbb{R}$).



cosh *x*

$\cosh x$ is not a one-one function. Consequently, in order to define its inverse we must restrict its domain to a part where it is one-one. For this purpose we chose that part of the domain where x is positive $(x \in \mathbb{R} \text{ such that } x \geq 0)$. On this region, which we call the principal value, $\cosh x$ is increasing and hence has an inverse

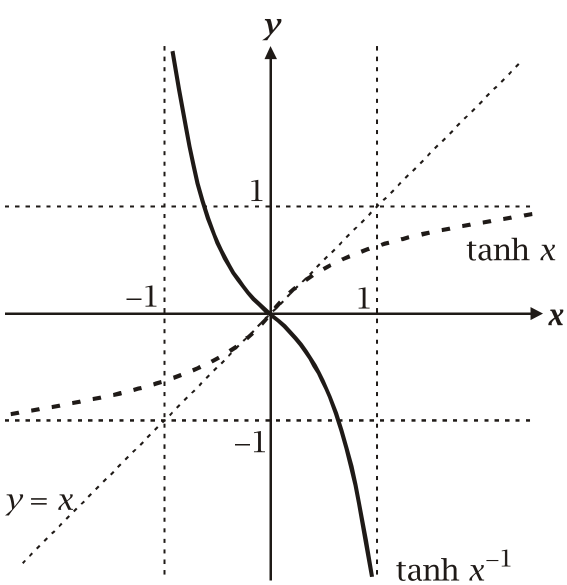
$$\cosh^{-1} x = \begin{Bmatrix} (1, \infty) \to \mathbb{R}^{+} \\ x \to \cosh^{-1} x \end{Bmatrix}$$



tanh *x*

tanh x in an always increasing, one-one function, that is asymptotic to $y = \pm 1$. As $x \to +\infty$ $\tanh x \to 1$ and as $x \to -\infty$ $\tanh x \to -1$. So the inverse is only defined on the interval $(-1,1)$.

$$\tanh^{-1} x = \begin{Bmatrix} (-1,1) \to \mathbb{R} \\ x \to \tanh^{-1} x \end{Bmatrix}$$



The Logarithmic forms of the inverse hyperbolic functions

The inverse hyperbolic functions can be expressed in logarithmic form.

$\sinh^{-1} x = \ln\left|x + \sqrt{x^2 + 1}\right|$

To show this, let $y = \sinh x$

Then

$$y = \frac{1}{2}\left(e^x - e^{-x}\right)$$
$$2y = e^x - e^{-x}$$
$$e^x - 2y - e^{-x} = 0$$
$$e^x\left(e^x - 2y - e^{-x}\right) = 0$$
$$e^{2x} - 2ye^x - 1 = 0$$

Which is a quadratic in e^x hence

$$e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$$
$$e^x = y \pm \sqrt{y^2 + 1}$$
$$x = \ln\left|y \pm \sqrt{y^2 + 1}\right|$$
$$f^{-1}(y) = \ln\left|y \pm \sqrt{y^2 + 1}\right|$$

That is

$$f^{-1}(x) = \ln\left|x \pm \sqrt{x^2 + 1}\right|$$

We can show similarly that

$\cosh^{-1} x = \ln\left|x + \sqrt{x^2 - 1}\right|, \quad x \geq 1$

$\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \quad |x| < 1$

Derivatives of the inverse hyperbolic functions

We state and prove the forms for the derivatives of the inverse hyperbolic functions.

$\mathbf{\sinh^{-1}x}$

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}}$$

Proof

Let $y = \sinh^{-1} x$

Then by taking the inverse

$x = \sinh y$

Then differentiating with respect to x:

$$\frac{d}{dx}\{x\} = \frac{d}{dx}\{\sinh y\}$$

$$\therefore 1 = \cosh y \cdot \frac{dy}{dx} \quad \text{by the chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cosh y}$$

Now $\cosh^2 y - \sinh^2 y = 1$

$$\therefore \cosh y = \sqrt{1 + \sinh^2 y}$$

But $\sinh y = x$; hence

$$\cosh y = \sqrt{1 + x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

$$\therefore \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}}$$

$\cosh^{-1} x$

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}$$

Proof

Let $y = \cosh^{-1} x$

Then $x = \cosh y$

$$\therefore \frac{dx}{dx} = \frac{d}{dx}\cosh y$$

$$1 = \sinh y \cdot \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sinh y}$$

But $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$

$$\therefore \frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}$$

$\tanh^{-1} x$

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1 - x^2}$$

Proof

Let $y = \tanh^{-1} x$

Then $x = \tanh y$

$\therefore$ On differentiating both sides with respect to x :

$$1 = \text{sech}^2 y \cdot \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\text{sech}^2 y}$$

Now $1 - \tanh^2 y = \text{sech}^2 y$

$\therefore \text{sech}^2 y = 1 - x^2$

$$\therefore \frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2}$$