Inverse of a 3 × 3 Matrix

Prerequisites

You should be familiar with the process of Gaussian row reduction and the related idea of reduction to echelon form.

Example (1)

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \\ -1 & 2 & 4 \end{pmatrix}$

Solution

The augmented matrix is

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -1 & 2 & 4 & 0 & 0 & 1 \end{array}\right)$$

Performing row reduction on this

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 2 & 0 & 5 & 0 & 1 & 1 \\ -2 & 4 & 8 & 0 & 0 & 2 \end{array}\right) \qquad \begin{array}{c} \\ (2)+(3) \\ (3)\times 2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & -6 & 1 & -1 & -1 \\ 2 & 0 & 5 & 0 & 1 & 1 \\ 0 & 5 & 7 & 1 & 0 & 2 \end{array}\right) \qquad \begin{array}{c} (1)-(2) \\ (1)+(3) \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 0 & 5 & -30 & 5 & -5 & -5 \\ 2 & 0 & 5 & 0 & 1 & 1 \\ 0 & 5 & 7 & 1 & 0 & 2 \end{array}\right) \qquad (1) \times 5$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & -37 & 4 & -5 & -7 \\ 2 & 0 & 5 & 0 & 1 & 1 \\ 0 & 5 & 7 & 1 & 0 & 2 \end{array}\right) \qquad (1)-(3)$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & -5 & \frac{20}{37} & -\frac{25}{37} & -\frac{35}{37} \\ 2 & 0 & 5 & 0 & 1 & 1 \\ 0 & 5 & 7 & 1 & 0 & 2 \end{array}\right) \qquad (1)\times\frac{5}{37}$$

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$$\left(\begin{array}{ccc|ccc} 0 & 0 & -7 & \frac{28}{37} & -\frac{35}{37} & -\frac{49}{37} \\ 2 & 0 & 0 & -\frac{20}{37} & \frac{12}{37} & \frac{2}{37} \\ 0 & 5 & 7 & 1 & 0 & 2 \end{array}\right) \qquad \begin{array}{l} (1)\times\frac{7}{5} \\ (2)+(1) \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & -1 & \frac{4}{37} & -\frac{5}{37} & -\frac{7}{35} \\ 1 & 0 & 0 & -\frac{10}{37} & \frac{6}{37} & \frac{1}{37} \\ 0 & 5 & 0 & \frac{65}{37} & -\frac{35}{37} & \frac{25}{37} \end{array}\right) \qquad \begin{array}{l} \\ \\ (3)+(1) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & {}^{-4}\!/_{37} & {}^{5}\!/_{37} & {}^{7}\!/_{37} \\ 1 & 0 & 0 & {}^{10}\!/_{37} & {}^{6}\!/_{37} & {}^{1}\!/_{37} \\ 0 & 1 & 0 & {}^{13}\!/_{37} & {}^{-7}\!/_{37} & {}^{5}\!/_{37} \end{array}\right) \qquad \begin{array}{l} (1)\times -1 \\ \\ \\ \end{array}$$

Swapping the rows around

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & {}^{10}\!/_{37} & {}^{6}\!/_{37} & {}^{1}\!/_{37} \\ 0 & 1 & 0 & {}^{13}\!/_{37} & {}^{-7}\!/_{37} & {}^{5}\!/_{37} \\ 0 & 0 & 1 & {}^{-4}\!/_{37} & {}^{5}\!/_{37} & {}^{7}\!/_{37} \end{array}\right)$$

$$\therefore\ A^{-1} = \frac{1}{37}\begin{pmatrix} 10 & 6 & 1 \\ 13 & -7 & 5 \\ -4 & 5 & 7 \end{pmatrix}$$

Formula for an inverse of a 3×3 non-singular matrix

We will now introduce a formula for the inverse of a 3×3 matrix. To employ this formula you must understand the symbol for the determinant. This is

If $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ then $\det A = \begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - rq$

Also for a 3×3 matrix, if $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b\begin{vmatrix} f & d \\ i & g \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

We now define the *adjugate* or *adjoint* of the matrix A to be

$$\operatorname{adj} A = \begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} h & i \\ b & c \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \begin{vmatrix} f & d \\ i & g \end{vmatrix} & \begin{vmatrix} i & g \\ c & a \end{vmatrix} & \begin{vmatrix} c & a \\ f & d \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} g & h \\ a & b \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

Using these concepts the formula for A^{-1} is

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$$

Example (1)

Using the definition of the adjugate matrix find the inverse of

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \\ -1 & 2 & 4 \end{pmatrix}$$

Solution

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} \\ \\ \begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} \\ \\ \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \end{pmatrix}$$

$$= \frac{1}{\det A} \begin{pmatrix} -10 & -6 & -1 \\ -13 & 7 & -5 \\ 4 & -5 & -7 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \\ -1 & 2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & 1 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix}$$

$$= 2(-10) - 13 - (6 - 2)$$

$$= -20 - 13 - 4$$

$$= -37$$

Hence

$$A^{-1} = -\tfrac{1}{37}\begin{pmatrix} 10 & 6 & 1 \\ -13 & 7 & -5 \\ 4 & -5 & -7 \end{pmatrix}$$

$$= \tfrac{1}{37}\begin{pmatrix} 10 & 6 & 1 \\ -13 & 7 & -5 \\ 4 & -5 & -7 \end{pmatrix}$$

It will be helpful to outline the proof that A^{-1} is indeed the inverse of A.. To show this we must demonstrate that

$$\mathbf{I} = AA^{-1} = \frac{1}{\det A}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} h & i \\ b & c \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \begin{vmatrix} f & d \\ i & g \end{vmatrix} & \begin{vmatrix} i & g \\ c & a \end{vmatrix} & \begin{vmatrix} c & a \\ f & d \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} g & h \\ a & b \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

To show $\mathbf{I} = AA^{-1}$ we have to multiply the matrices on the left and evaluate each term. We shall just indicate how this is done. Let us mark the entries of AA^{-1} as follows, and consider each entry separately

Let $AA^{-1} = \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix}$

That is

$$AA^{-1} = \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} h & i \\ b & c \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \begin{vmatrix} f & d \\ i & g \end{vmatrix} & \begin{vmatrix} i & g \\ c & a \end{vmatrix} & \begin{vmatrix} c & a \\ f & d \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} g & h \\ a & b \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

Then by matrix multiplication

$$t_{1,1} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b\begin{vmatrix} f & d \\ i & g \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = \det A$$

Similarly, $t_{2,2} = t_{3,3} = \det A$. For the other entries

$$\begin{aligned} t_{1,2} &= a\begin{vmatrix} h & i \\ b & c \end{vmatrix} + b\begin{vmatrix} i & g \\ c & a \end{vmatrix} + c\begin{vmatrix} g & h \\ a & b \end{vmatrix} \\ &= a\,(hc - ib) + b(ia - gc) + c(gb - ah) \\ &= ach - abi + abi - bcg + bcg - ach \\ &= 0 \end{aligned}$$

Similarly we could show that all the other non-diagonal entries evaluate to 0. Hence

$$AA^{-1} = \frac{1}{\det A}\begin{pmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{pmatrix} = \mathbf{I}$$

So A^{-1} is indeed the inverse of A. Looking at how the formula is constructed, let us note two properties of matrices and determinants.

1. If any two rows or columns of a non-singular matrix A are interchanged to give a matrix A' then $\det \mathrm{A}' = -\det A$. Interchanging the rows or columns of a matrix has the effect of multiplying the determinant by –1.

2. Cyclically permuting the rows or columns of a matrix A does not alter the determinant of A.

To illustrate the first property, take the matrix we have been studying above

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \\ -1 & 2 & 4 \end{pmatrix}$$

with $\det A = -37$. Interchanging rows (1) and (2) gives

$$A' = \begin{pmatrix} 3 & -2 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & 4 \end{pmatrix}$$

with $\det A' = -\det A = 37$

To illustrate the cyclical permutation property let us permute the rows of A as follows: $(1) \to (3)$; $(3) \to (2)$; $(2) \to (1)$ to give

$$B = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 4 \\ 3 & -2 & 1 \end{pmatrix}$$

The determinant is unchanged: $\det B = \det A = -37$. If we recall our proof that A^{-1} was the inverse of A, note that the $t_{1,1}$ entry is $\det A$ where $\det A$ is evaluated as follows.

Given $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$t_{1,1} = \det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b\begin{vmatrix} f & d \\ i & g \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The determinants $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$, $\begin{vmatrix} f & d \\ i & g \end{vmatrix}$, $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$ are the entries in the first column of A^{-1}.

Then, cyclically permuting the rows of A we obtain

$$A = \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$$

$$t_{2,2} = \det A = d\begin{vmatrix} h & i \\ b & c \end{vmatrix} + e\begin{vmatrix} i & g \\ c & a \end{vmatrix} + f\begin{vmatrix} g & h \\ a & b \end{vmatrix}$$

The determinants $\begin{vmatrix} h & i \\ b & c \end{vmatrix}$, $\begin{vmatrix} i & g \\ c & a \end{vmatrix}$ and $\begin{vmatrix} g & h \\ a & b \end{vmatrix}$ are the entries in the second column of A^{-1}.

Similarly, cyclically permuting A again gives

$$A = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} \qquad t_{3,3} = \det A = g\begin{vmatrix} b & c \\ e & f \end{vmatrix} + h\begin{vmatrix} c & a \\ f & d \end{vmatrix} + i\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

The determinants $\begin{vmatrix} b & c \\ e & f \end{vmatrix}, \begin{vmatrix} c & a \\ f & d \end{vmatrix}, \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ are the entries in the third column of A^{-1}.

Thus A^{-1} is given by

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} \text{determinants} & \text{determinants} & \text{determinants} \\ \text{derived from} & \text{derived from} & \text{derived from} \\ \text{first cyclic} & \text{second cyclic} & \text{third cyclic} \\ \text{permutatuion} & \text{permutatuion} & \text{permutatuion} \\ \text{of A} & \text{of A} & \text{of A} \end{pmatrix}$$

$$= \frac{1}{\det A}\begin{pmatrix} \text{determinants from} & \text{determinants from} & \text{determinants from} \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix} & \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{\det A}\begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} h & i \\ b & c \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \begin{vmatrix} f & d \\ i & g \end{vmatrix} & \begin{vmatrix} i & g \\ c & a \end{vmatrix} & \begin{vmatrix} c & a \\ f & d \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} g & h \\ a & b \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

Definition of the adjugate

There is a formal definition of the adjugate that can be illustrated as follows. Take the matrix A.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Delete from it the 1st column and the 1st row to obtain the determinant

$$\begin{vmatrix} e & f \\ h & i \end{vmatrix}$$

this is the $A_{1,1}$ entry in the adjugate. Now delete from A the 2nd column and the 1st row to obtain the determinant

$$\begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

Since the row number (1) + the column number (2) = 3 is odd, then multiply this by –1. So obtain

$$-\begin{vmatrix} d & f \\ g & i \end{vmatrix} = \begin{vmatrix} f & d \\ i & g \end{vmatrix}$$

This determinant is called the *cofactor* of the $a_{1,2}$ entry in A and is the $A_{1,2}$ entry in the adjugate. Note that the suffixes have been swapped around. The 2nd column, 1st row in A gives the entry of the 1st column, 2nd row in adj A. Continue in this way. Delete the ith row and the jth column.

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Multiply by –1 if $i + j$ is an odd number. The resultant determinant is the entry of the *j*th column and *i*th row in adj *A*. Another way of doing this is to form the matrix of the cofactors of *A* and the adjugate is the transpose of this matrix – that is, when the rows and columns a swapped over.

Example (2)

Let $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ -2 & 1 & 2 \end{pmatrix}$

Find (a) $\det A$, (b) the matrix of cofactors of *A*, (c) the adjugate of *A*, (d) the inverse of *A*, (e) verify by multiplying AA^{-1} that the inverse of *A* found in part (d) is indeed the inverse.

Solution

$$\det A = \left\| \begin{matrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ -2 & 1 & 2 \end{matrix} \right\| = 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} = 3 + 4 + 8 = 15$$

To find the matrix of cofactors first let us write out the signs of the cofactors of *A* according to the rule that for the *i* column and *j* row, if $i + j$ is odd, then the cofactor will be multiplied by –1. This gives the following pattern of determinants in the cofactor matrix of *A*.

$$C = \begin{pmatrix} +\begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} & +\begin{vmatrix} & \\ & \end{vmatrix} \\ -\begin{vmatrix} & \\ & \end{vmatrix} & +\begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} \\ +\begin{vmatrix} & \\ & \end{vmatrix} & -\begin{vmatrix} & \\ & \end{vmatrix} & +\begin{vmatrix} & \\ & \end{vmatrix} \end{pmatrix}$$

This pattern of alternating signs will always apply for every 3×3 matrix, so it is useful starting point. Please note, there are 9 entries here, and the pattern of alternating signs does not mean that these determinants will be added together. Now we fill in the determinants. Let us illustrate how this is done for the second row of determinants. For this row we delete the 2nd row from *A* to get

$$\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \end{pmatrix}$$

Now deleting the 1st, 2nd and 3rd columns successively gives the three cofactors of the second row of *C*.

$$\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \qquad \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}$$

Fill in all the other determinants in the same way and then evaluate them

$$C = \begin{pmatrix} +\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \\ +\begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & 4 \\ 6 & 6 & 3 \\ 0 & 5 & 5 \end{pmatrix}$$

Then

$$\operatorname{adj} A = C^T = \begin{pmatrix} 3 & 6 & 0 \\ -2 & 6 & 5 \\ 4 & 3 & 5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$$= \frac{1}{15}\begin{pmatrix} 3 & 6 & 0 \\ -2 & 6 & 5 \\ 4 & 3 & 5 \end{pmatrix}$$

Finally

$$AA^{-1} = \frac{1}{15}\begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 6 & 0 \\ -2 & 6 & 5 \\ 4 & 3 & 5 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{pmatrix} = \mathbf{I}$$

In fact here we have not given the formal definition of the adjugate because our concern is with finding the inverse of a 3×3 matrix. The definition of the adjugate can be extended along these lines to cover square matrices of any size.