Inverse Trigonometric Functions

Prerequisites

(1) Finding an angle using the inverse of sine, cosine and tangent

You should be already familiar with the process of finding the inverse of a trigonometric ratio. That is, for example, given the sine of an angle, you should be able to find the angle at least for angles less than 90°.

Example (1)

Find θ if $\sin\theta = 0.4$

Solution

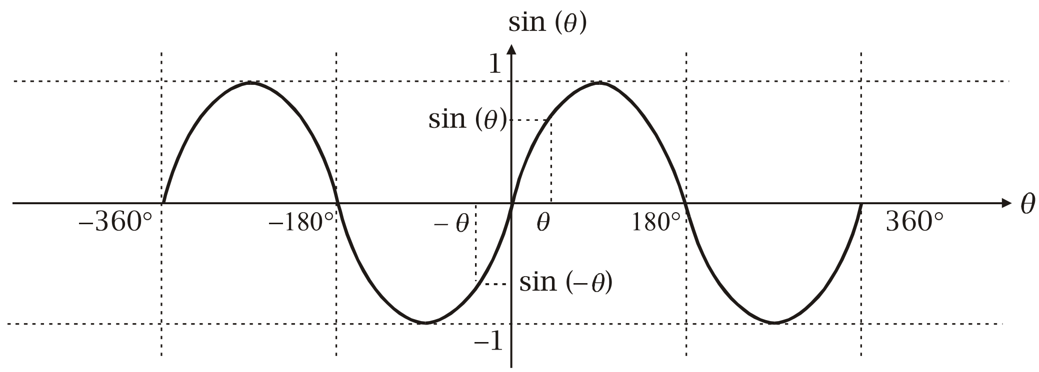
$\sin\theta = 0.4$

$\theta = \sin^{-1}(0.4) = 23.6° \ (0.1°)$

The expression $\sin^{-1}(0.4)$ is read "inverse sine of 0.4".

(2) Trigonometric functions

You should be aware that we can define trigonometric functions that hold for all values of θ. For example, the function $f(\theta) = \sin\theta$ has the following graph.



(3) Domains and co-domains

You should be aware that a *function* is a *mapping* from one set (called the *domain*) to another set (called the *co-domain*). That is, a function is a rule taking you from one number to another. For a given application of a rule, the number in the domain is called the *argument* of the function and the number to which it is mapped by the rule is called its *value*.

Example (2)

State the domain and co-domain of the functions

$f(\theta) = \sin\theta \qquad f(\theta) = \cos\theta \qquad f(\theta) = \tan\theta$

Solution

$f(\theta) = \sin\theta$

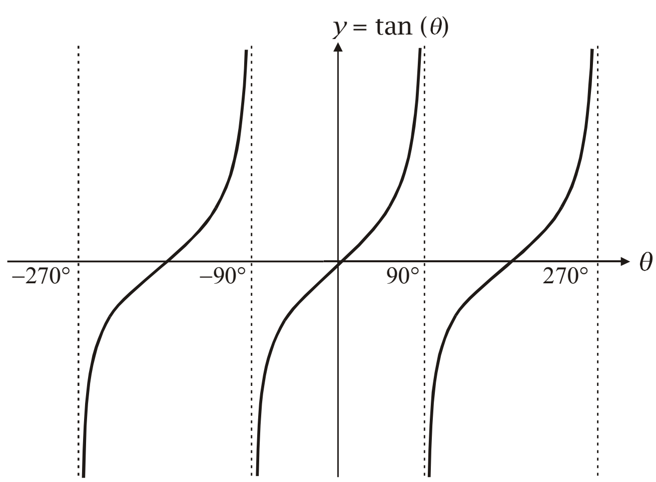
From the graph of $f(\theta) = \sin\theta$ given above we can see that $\sin\theta$ is defined for all values of θ, so the domain is the set ℝ, the set of all real numbers. However, $f(\theta) = \sin\theta$ takes only values between +1 and −1, so the co-domain is $-1 \le x \le 1$.

$f(\theta) = \cos\theta$

As for $\sin\theta$. Domain ℝ. Co-domain $-1 \le x \le 1$.

$f(\theta) = \tan\theta$

The graph of $f(\theta) = \tan\theta$ is

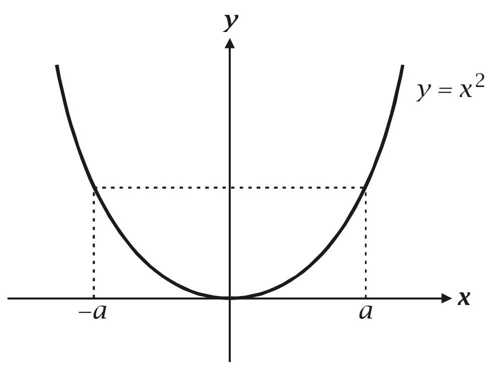


This shows that $f(\theta) = \tan\theta$ is periodically undefined. It has asymptotes to the lines $\theta = 90° \pm n180°$ where n is any integer positive or negative. So the domain of $f(\theta) = \tan\theta$ is the whole of $\mathbb{R}$ less the points where it is undefined – called *singularities*, which are where $\theta = 90° \pm n180°$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Values of $\tan\theta$ are not restricted and the co-domain is the whole of $\mathbb{R}$.

Inverses of functions

If a function *f* maps *x* to *y*, then the inverse of that function, written f^{-1}, maps *y* to *x*. The inverse of a function reverses the process represented by that function. However, not all functions have inverses. This is because functions can be *many-one* or *one-one*. An example of a many-one function is

$$f\begin{cases}\mathbb{R} \to \mathbb{R} \\ x \to x^2\end{cases}$$



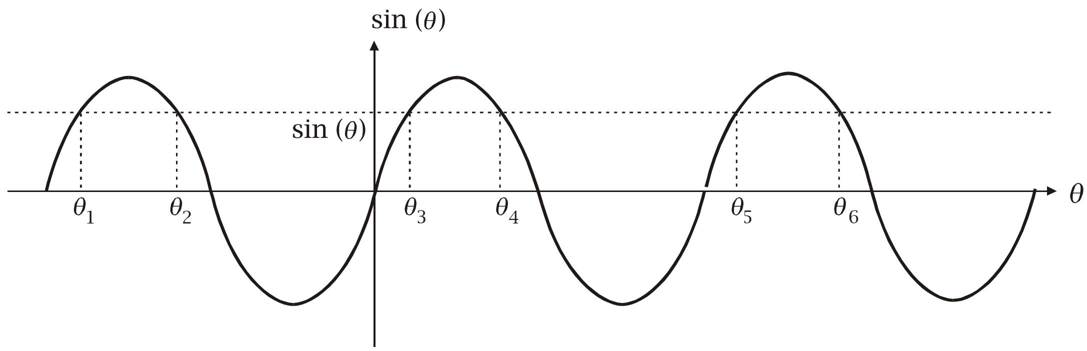
This is many-one because there are two values in the domain giving the same value in the co-domain: $f(a) = f(-a)$. A many-one function is a function such that there are two or more arguments in the domain giving the same value in the co-domain. A many-one function cannot have an inverse because the arguments of the "inverse" would have more than one value, and a function must specify just one value for each argument. A one-one function specifies for each argument just one value. For a function to have an inverse it must be one-one. A one-one function is either always increasing or always decreasing. An always-increasing function is also called a *monotone increasing* function, and an always-decreasing function is also called a *monotone decreasing* function.

Example (3)

Determine whether the trigonometric functions are one-one functions.

Solution

All the trigonometric functions are periodic functions and are consequently not one-one. For example, $f(\theta) = \sin\theta$ is sometimes increasing and sometimes decreasing.



The graph shows that for a given value of $f(\theta) = \sin\theta$ there are many values of θ such that $f(\theta) = \sin\theta$. The graph shows six of these, but as $f(\theta) = \sin\theta$ is periodic there are an infinite number of solutions.

Example (3) continued

In example (1) we argued as follows

$\sin\theta = 0.4$

$\theta = \sin^{-1}(0.4) = 23.6° \ (0.1°)$

We called $\sin^{-1}(0.4)$ the "inverse sine of 0.4". Since $f(\theta) = \sin\theta$ is not a one-one function what has happened to enable us to find its inverse?

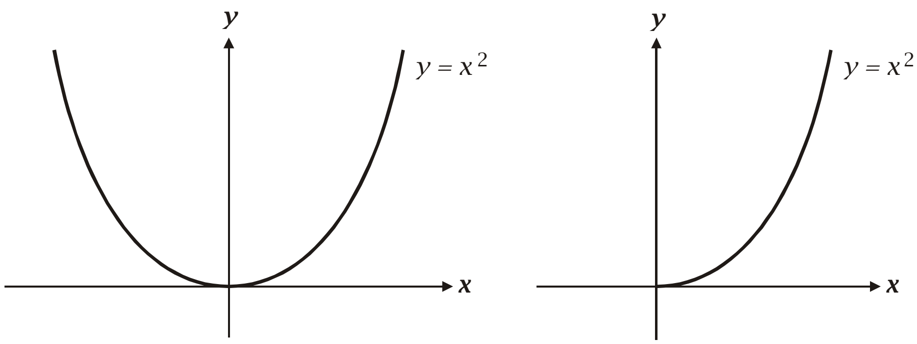
Solution

$f(\theta) = \sin\theta$ is *not* a one-one function so strictly $\sin^{-1}(0.4)$ has many solutions. When we argue $\theta = \sin^{-1}(0.4) = 23.6° \ (0.1°)$ we have chosen *just one* of these many solutions. We have *restricted* our attention to just one solution.

This "trick" of restricting solutions of many-one functions to just one of those solutions is the key to finding the inverses of many-one functions. You have been using the inverse of sine, cosine and tangent for some while now, and this rule that $\sin^{-1}(x)$ finds just one of the possible solutions has been implicit in what you have been doing. Now we make that process explicit. Furthermore, we wish to use our knowledge of the periodic nature of trigonometric functions to find all the solutions to an equation such as $y = \sin^{-1}(x)$.

Creating one-one functions by restriction

A one-one function can be created from a many-one function by restricting the domain. This means diminishing the size of the domain so that the function becomes increasing or decreasing on the reduced domain. For example



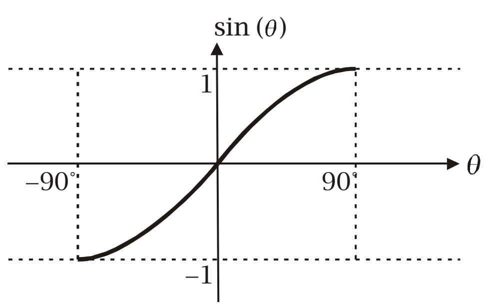
$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \to x^2 \end{cases} \qquad \xrightarrow[\text{RESTRICTTION}]{} \qquad g : \begin{cases} \mathbb{R}^+ \to \mathbb{R}^+ \\ x \to x^2 \end{cases}$$

The domain is restricted to contain only positive real numbers. The inverse of this restricted function $g(x) = x^2$ is called the square root.

$$g^{-1} : \begin{cases} \mathbb{R}^+ \to \mathbb{R}^+ \\ x \to \sqrt{x} \end{cases}$$

In order to specify the inverse of a trigonometric function it is necessary to restrict the domain of the function so that the restricted part becomes one-one. We call this restriction, when it takes a standard form, the *principal domain*. The standard restrictions of trigonometric functions are as follows.

(**1**) The principle domain of $y = \sin\theta$ is the closed interval $-90° \le x \le 90°$.



What this means is that whenever we calculate the inverse of the sine of some angle, the angle we obtain will lie in the interval $-90° \le \theta \le 90°$. With $y = \sin\theta$ confined to the domain $-90° \le \theta \le 90°$ in this way, the function is always increasing, that is one-one, and hence has an inverse, $\theta = \sin^{-1} y$.

Example (4)

Define the principle domain of $y = \sin\theta$ using radian measure.

Solution

In radians the principle domain is $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

The solution to $\theta = \sin^{-1} y$ where θ has been restricted to the principle domain is called the principle value.

Example (5)

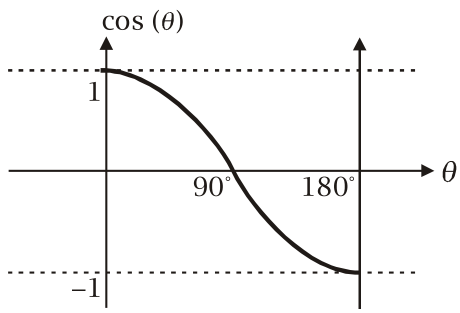
Find in both degrees and radians the principle value of $\sin^{-1}(-0.5)$

Solution

In degrees $\sin^{-1}(-0.5) = -30°$ In radians $\sin^{-1}(-0.5) = -\frac{\pi}{6}$

The angle $-30°$ is equivalent to the angle $330°$. When you use a calculator to find the inverse sine of an angle you will obtain an answer between –90° and 90°, but you will have to decide whether this is *the* answer you want, or whether another angle is appropriate.

(2) The principle domain of $y = \cos\theta$ is the closed interval $0 \le \theta \le 180°$.



So when calculating the inverse of a cosine, you will obtain a value that lies in the interval $0 \le \theta \le 180°$ (in radians this is $0 \le \theta \le \pi$).

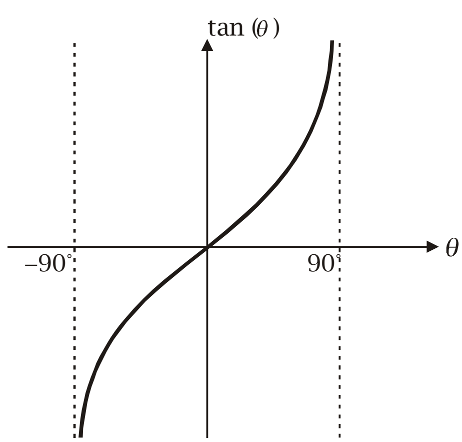
Example (6)

Find in radians the principle value of $\cos^{-1}\frac{1}{\sqrt{2}}$. Write down two other angles differing from this by $\pm 2\pi$ radians that also have same value of $y = \cos\theta$. Give your answer in radians.

Solution

You should be aware from your knowledge of special triangles that if $\theta = \cos^{-1}\frac{1}{\sqrt{2}}$ then $x = \frac{\pi}{4}$ radians, (This is the same as the angle 45°). Two other angles, θ_1 and θ_2, that also have $\cos\theta = \frac{1}{\sqrt{2}}$ are $\frac{\pi}{4} + 2\pi = \frac{9}{4}\pi$ (405°) and $\frac{\pi}{4} - 2\pi = -\frac{7}{4}\pi$ (–315°).

(3) The principle domain of $y = \tan\theta$ is the *open* interval $-90° < x < 90°$. (In radians $-\frac{\pi}{2} < x < \frac{\pi}{2}$.) It is an open interval because the arguments $-90°$ and $90°$ are *not* included in the domain. The term *closed* interval means that the end-points are included in the domain, whereas an "open interval" does not include the end-points.



Using the language of functions we can write the inverse functions with their domains as follows.

$$\sin^{-1}\begin{cases} x \to \sin^{-1}(x) \\ [-1,1] \to \left[-\pi/2, \pi/2\right] \end{cases}$$

$$\cos^{-1}\begin{cases} x \to \cos^{-1}(x) \\ [-1,1] \to [0,\pi] \end{cases}$$

$$\tan^{-1}\begin{cases} x \to \tan^{-1}(x) \\ \mathbb{R} \to \left(-\pi/2, \pi/2\right) \end{cases}$$

In these expressions we have deliberately switched from using θ to x as the variable – to remind you that any letter can be used to represent a variable. The use of either θ or x to represent an angle is common.

Example (7)
Write the equivalents of the above definitions of the inverse functions in degrees.

Solution

$$\sin^{-1}\begin{cases} x \to \sin^{-1}(x) \\ [-1,1] \to [-90°,90°] \end{cases}$$

$$\cos^{-1}\begin{cases} x \to \cos^{-1}(x) \\ [-1,1] \to [0,180°] \end{cases}$$

$$\tan^{-1}\begin{cases} x \to \tan^{-1}(x) \\ \mathbb{R} \to (-90°,90°) \end{cases}$$

Solutions to trigonometric equations

We wish to solve *trigonometric equations* of the form

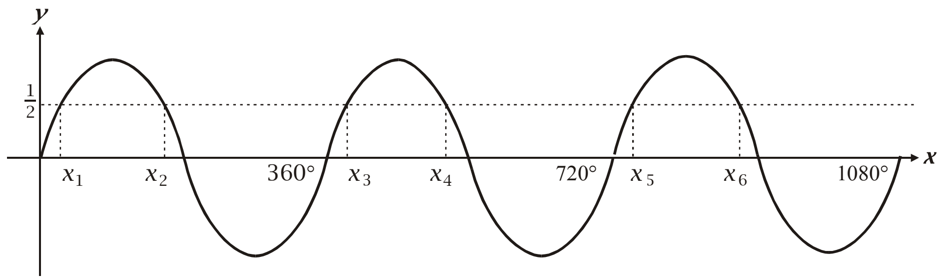
$\sin(kx) = c$

where k and c are real numbers. This will require the use of inverse trigonometric functions. The calculator will provide the principle value, but you must use your knowledge of the periodic nature of this function to find as many of the other values as required.

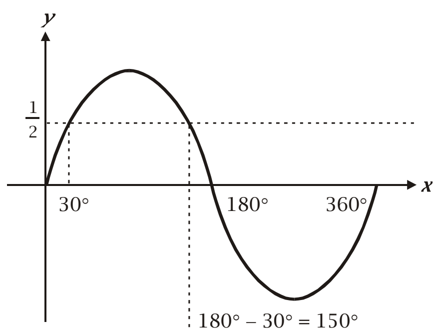
Example (8)
Solve $\sin x = \frac{1}{2}$ for values of x in the interval $0 \le x \le 720°$.

Solution
This question has asked for the answer in degrees. To solve the problem begin by sketching the graph of $y = \sin x$. On to this graph draw a horizontal line representing the

value $y = \sin x = \frac{1}{2}$. The function $y = \sin x$ is periodic with period 2π. Solutions to the equation $\sin x = \frac{1}{2}$ are intersections of this function with the line $y = \frac{1}{2}$.



The diagram shows that because $y = \sin x$ is a periodic function there are an infinite number of values of the angle x such that $\sin x = \frac{1}{2}$. However, in the interval $0 \le x \le 720°$ there are just four, and these are marked x_1, x_2, x_3 and x_4 in the diagram. The principal value of x for which $\sin x = \frac{1}{2}$ is $x = \sin^{-1}\left(\frac{1}{2}\right) = 30°$. Symmetry shows that the other angle in the interval $0 \le x \le 360°$ is 150°.



The other two solutions are found by adding 360° to these.

$x_1 = 30°,\ x_2 = 150°,\ x_3 = 390°,\ x_4 = 510°$

Example (9)

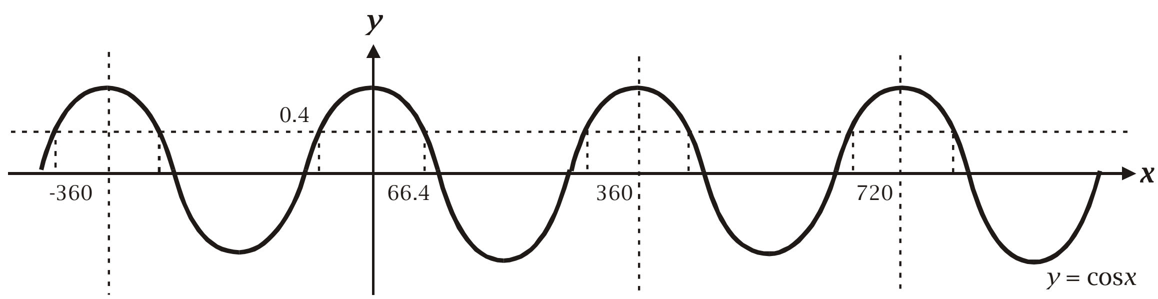
Solve $\cos(2x) = 0.4$ for $0 \le x \le 360°$.

Solution

$\cos(2x) = 0.4$

$2x = \cos^{-1}(0.4)$

We begin by sketching a graph of $\cos x$ and the intersection of this graph with the line $y = 0.4$.



The graph shows that there are an infinite number of solutions to the equation

$x = \cos^{-1}(0.4)$

since $y = \cos x$ is a periodic function. The solution in the principle domain is

$\cos^{-1}(0.4) = 66.4° \ (0.1°)$

Another solution is $-66.4° \ (0.1°)$. The other solutions are found by adding multiples of $360°$ to these

293.6°, 426.4°, 653.6° ....

The question asked for the solutions to $2x = \cos^{-1}(0.4)$, so we have

$2x = 66.4°, \ 293.6°, 426.4°, 653.6°$ ....

Hence

$x = \dfrac{66.4°}{2}, \ \dfrac{293.6°}{2}, \ \dfrac{426.4°}{2}, \ \dfrac{653.6°}{2}$ ....

$x = 33.2°, \ 146.8°, 213.2°, 326.8°$ ....

The solutions in the interval $0 \le x \le 360°$ are

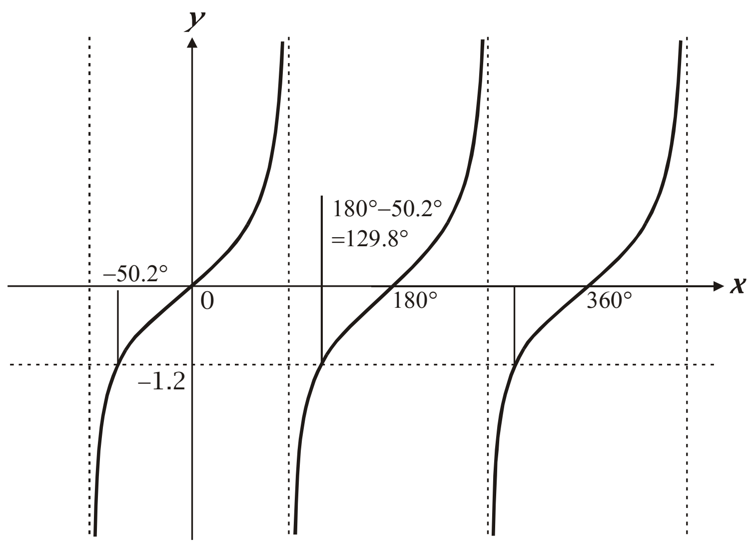
$x = 33.2°, \ 146.8°, 213.2°, 326.8° \ (0.1°)$

(We have removed the dots that indicated that the solutions were infinite decimals and taken an approximation to the nearest 0.1°. There are infinite solutions to $2x = \cos^{-1}(0.4)$, but a finite number in the interval $0 \le x \le 360°$.)

The presence here of the annotation makes the whole process look more complicated than it is. We give the solution to the next example without annotation.

Example (10)
Find all the values of x in the range 0° to 180° satisfying
$\tan 3x = -1.2$

Solution
$\tan 3x = -1.2$
$3x = \tan^{-1}(-1.2)$



$3x = -50.1944... \pm 180n$
$\quad = ...,129.8, 309.8, 489.8, 669.8, 849.8,...$

$x = 43.3°, 103.3°, 163.3°$ where $0° < x < 180°$ $(0.1°)$

Quadratic trigonometric equations

The meaning of the expression $\cos^2 x$ is given by

$\cos^2 x = (\cos x)^2 = (\cos x) \times (\cos x)$

In other words $\cos^2 x$ is shorthand for $(\cos x)^2$. The expression $\cos^2 x$ is read "cos squared x".

The expression $(\cos x)^2$ is read "cos x all squared", but they mean the same thing. It is a device for avoiding writing too many brackets into an expression.

An equation of the form
$2\cos^2 x - \cos x - 1 = 0$
is a quadratic equation in x and can be solved in the way quadratic equations are solved.

Example (11)

Find all values of x in the range $0° \le x \le 360°$ satisfying

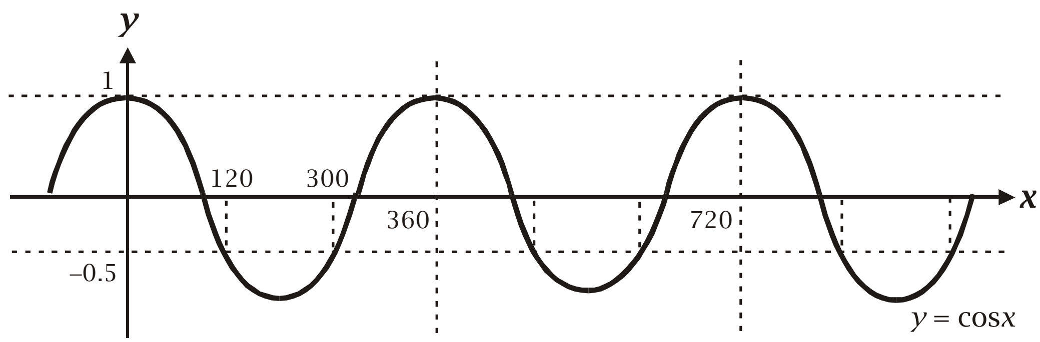
$2\cos^2 x - \cos x - 1 = 0$

Solution

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = -\frac{1}{2}$ or $\cos x = 1$



In the range $0° \le x \le 360°$

$\cos x = -\frac{1}{2} \quad \Rightarrow \quad x = 120°$ or $x = 300°$

$\cos x = 1 \quad \Rightarrow \quad x = 0°$ or $x = 360°$

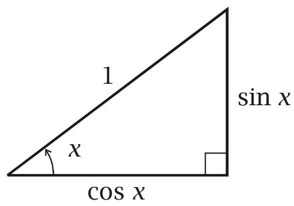
The four solutions are $x = 0°$, $x = 120°$, $x = 300°$, $x = 360°$.

Disguised quadratic trigonometric equations

The equation

$3\sin^2 x - 2\cos x - 2 = 0$

appears to be of a different type to the one we met in the last section. It involves expressions for *both* $\sin x$ and $\cos x$. However, it is possible to replace expressions involving $\sin^2 x$ by expressions involving $\cos^2 x$ and vice-versa. To see how consider the following triangle.



This is a right-angled triangle with hypotenuse length 1. The adjacent side has length $\cos x$ and the opposite side $\sin x$, which can be seen for instance from the equation

$$\cos x = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{adjacent}}{1} = \text{adjacent}.$$

Then from this triangle, by Pythagoras's theorem

$$\sin^2 x + \cos^2 x = 1$$

Rearrangement of this gives either

$$\sin^2 x = 1 - \cos^2 x \quad \text{or} \quad \cos^2 x = 1 - \sin^2 x$$

This means we can replace an expression involving $\sin^2 x$ by one involving $1 - \cos^2 x$ and we can replace an expression involving $\cos^2 x$ by one involving $1 - \sin^2 x$.

Example (12)

Find all values of x in the range $0° \le x \le 360°$ satisfying

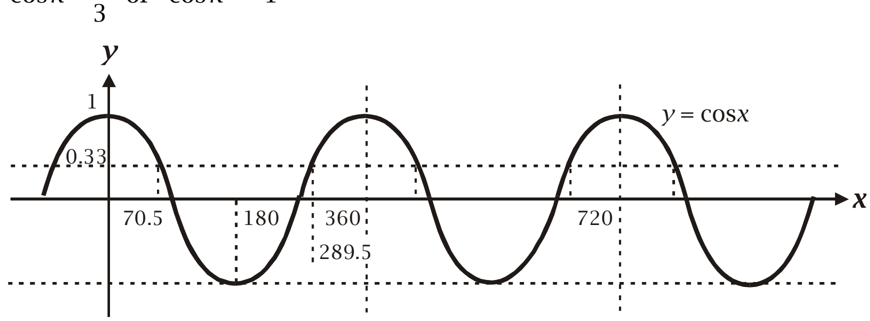
$$3\sin^2 x - 2\cos x - 2 = 0$$

Solution

$$3\sin^2 x - 2\cos x - 2 = 0$$

Substituting $\sin^2 x = 1 - \cos^2 x$

$$3\left(1 - \cos^2 x\right) - 2\cos x - 2 = 0$$
$$3 - 3\cos^2 x - 2\cos x - 2 = 0$$
$$3\cos^2 x + 2\cos x - 1 = 0$$
$$(3\cos x - 1)(\cos x + 1) = 0$$
$$\cos x = \frac{1}{3} \quad \text{or} \quad \cos x = -1$$



$$x = 70.5°, 180°, 289.5° \quad \text{where } 0° \le x \le 360° \quad (0.1°)$$