Isomorphisms

The whole motivation for the study of abstract structures is in order to encapsulate what is common to several mathematical objects. Mathematicians study structure because the solution to a problem concerning one structure can be generalised to all other similar structures. Central to this idea of structure, then, is the concept of identity of structures. In the context of group theory we need to be able to demonstrate when two groups are essentially the same. In order to do this we will define the concept of an isomorphism. An isomorphism, ϕ is a structure preserving mapping (a mapping that maintains the structure).

1. It maps the elements of one set *G* to elements of the other. In order to preserve structure the size of *G* and *H* must be identical. That is, an isomorphism is a one-one mapping (a bijection), of *G* to *H*.

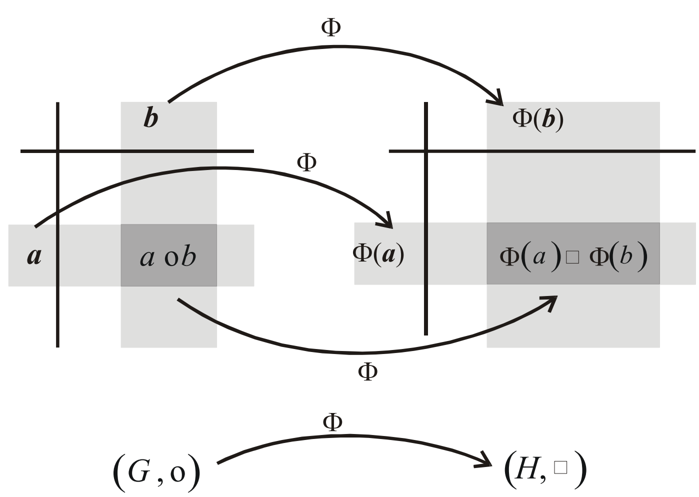
2. The binary operation O on *G* corresponds under the isomorphism to the binary operation □ on *H*.

3. The isomorphism preserves the structure in *G* and shows that this structure is replicated in *H*. To do this we show that

   $$\Phi(a \circ b) = \Phi(a) \square \Phi(b)$$

   where φ is the isomorphism, a, b are elements of *G*, and $\phi(a),\ \phi(b)$ are the corresponding elements of H.

   This third property can be illustrated thus



The isomorphism Φ takes an element from the group *G* and maps it to an elment in the group *H*

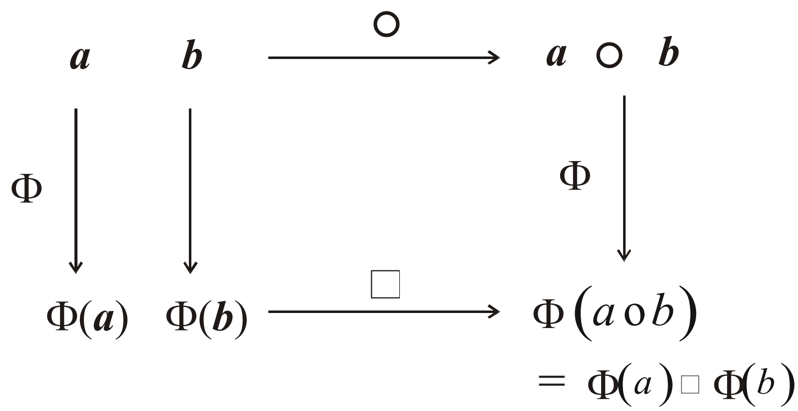
There are two ways to navigate between the element, $a \circ b$ of the group $(G, \circ)$ and its corresponding element in the group $(H, \square)$.

1. Find $\phi(a)$ and $\phi(b)$ in H then find $\phi(a) \square \phi(b)$ in H

2. Find $a \circ b$ in G, then find $\phi(a \circ b)$ in H

What this property states is that both routes are equivalent. Whichever route you take you reach the same element in H.

The following diagram also illustrates this property



When two groups are isomorphic we use the symbol we express this with the symbol
$G \cong H$

or more explicity
$(G, \circ) \cong (H, \square)$

<u>Summary</u>

An isomorphism, ϕ, is a bijection (one-one correspondence) such that

1. ϕ: $(G, \circ) \to (H, \square)$

2. $\phi(a \circ b) = \phi(a) \square \phi(b)$ for all $a, b \in G$

Note, this second property is called the homomorphism property – it is the structure preserving mapping between the groups. We will learn subsequently that the structure can be preserved even if the mapping is not a bijection (one-one). When the structure is preserved but the mapping is not one-one, then the relationship established between the groups is a homomorphism.

When structures are not isomorphic

We can illustrate the difference of two structures by showing that there cannot be an isomorphism between them.
Consider the groups $\mathbb{Z}_4$ and **K**.

$\mathbb{Z}_4$ is the cyclic group of four elements.

K is the Klein group and is identical to the symmetry group of a rectangle.

The combination tables of $\mathbb{Z}_4$ and K are

+(mod 4)	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\square$	I	R	Q_X	Q_Y
I	I	R	Q_X	Q_Y
R	R	I	Q_Y	Q_X
Q_X	Q_X	Q_Y	I	R
Q_Y	Q_Y	Q_X	R	I

In **K** every element is its own inverse, but in $\mathbb{Z}_4$ only 0 and 2 are their own inverses. Therefore there cannot be a structure preserving isomorphism. More formally, we show that every possible mapping from $\mathbb{Z}_4$ to **K** does not satisfy the structure preserving property.

$$\phi(a \circ b) = \phi(a) \square \phi(b)$$

For example, suppose ϕ is the mapping

$0 \rightarrow \mathrm{I}$
$1 \rightarrow \mathrm{R}$
$2 \rightarrow \mathrm{O}_X$
$3 \rightarrow \mathrm{O}_Y$

Then, $\phi(1+3) = \phi(0) = \mathrm{I}$
but $\phi(1) \ \square \ \phi(3) = \mathrm{R} \ \square \ \mathrm{Q}_{\mathrm{X}} = \mathrm{Q}_{\mathrm{Y}}$

Hence, $\phi(1+3) \neq \phi(1) \ \square \ \phi(3)$

It would be possible to show a similar failure for every possible bijection from $\mathbb{Z}_4$ to *K*.

When two groups are isomorphic then their combination tables are such that one differs from the other only in (1) the labels used to designate the elements; (2) the order in which the rows (and columns) are set down. Therefore, in order to show that two groups are isomorphic it is sufficient to rearrange the combination table for one of them - by permuting rows - to demonstrate that they are the same structure because the elements follow each other in the same order.