Lagrange's Interpolating Polynomial

Langrange's Interpolating Polynomial

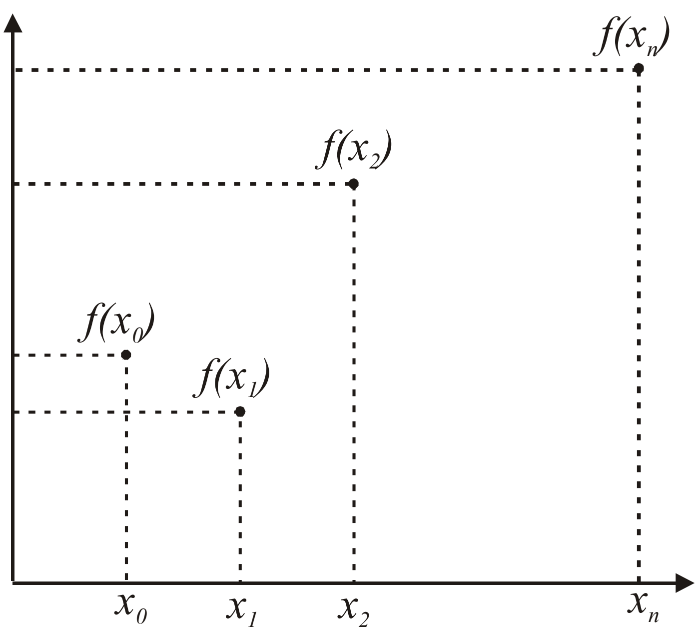
In many problems data is given as a number of points $x_0, x_1, x_2, \ldots x_n$ with their corresponding values

$y_0 = f(x_0)$

$y_1 = f(x_1)$

and so forth.

A diagrammatic representation could be

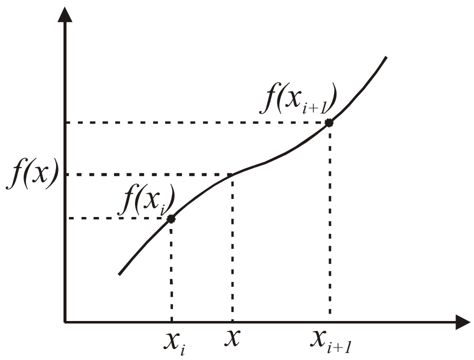


Here there are $n+1$ data points - that is $n+1$ ordered pairs (x_i, y_i)

It can be shown that there is just one polynomial of degree n that takes the desired value of f at each of the $n+1$ points $x_0, x_1, \ldots x_n$. If we are able to obtain this polynomial we can use it to approximate other values not originally given.

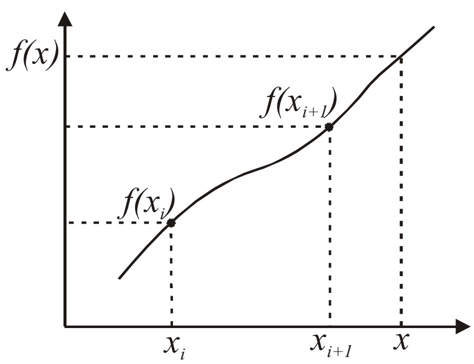
The polynomial, f, that has this property of taking the values $y_0, y_1, \ldots, y_n$ at points $x_0, x_1, \ldots x_n$ is called the Langrange Interpolating polynomial.

Where the point x, where the approximation is taken, lies between two values x_i and x_{i+1} the process of approximation is called interpolation.



Interpolation

Where the point being approximated lies outside the range of values, it is called extrapolation.



Extrapolation

Formula for the interpolating polynomial

Lagranges Interpolation Polynominals

The purpose of Lagrange's interpolation polynomial is to find a polynomial function of degree n passing through n points. In order to understand the formula you need to understand the symbols.

$\sum$ which means sum or add up.

$\prod$ which means find the product(multiply together)

The nth degree Lagrange interpolating polynomial is

$$p_n(x) = \sum_{k=0}^{n} a_k(x) y_k$$

where

$$a_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

The first part of the formula expands thus

$$p_n(x) = a_1(x) y_1 + a_2(x) y_2 + \cdots + a_n(x) y_n$$

The points through which are polynomial has to pass have co-ordinates $(x_1 y_1), (x_2 y_2)$ etc.

So in this formula you are multiplying for each point 1, 2, ….k….n a function $a_k(x)$ that is dependent on x by its corresponding y_k value.

$a_k(x)$ is a product and it expands as follows

$$a_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

In the numerator of this expression you subtract every x_i value systematically from x and multiply the whole lot together except that you leave x_k out. In the denominator you subtract every x_i value systematically from x_k and multiple the whole lot together, except that you leave x_k out.

Although the formula seems unduly complicated at first, it is in fact a straightforward mechanical job, a little more involved, perhaps, than using, say, the quadratic formula but in principle no more tricky. We now illustrate its use.

Example

Find Lagrange's interpolation polynominal through the points $(-3, 2), (-1, 4), (3, 1)$ and $(6, -2)$. Verify that it does pass through the given points.

Solution

$x_0 = -3, \qquad x_1 = -1, \qquad x_2 = 3, \qquad x_3 = 6$

$y_0 = 2, \qquad y_1 = 4\ , \qquad y_2 = 1, \qquad y_3 = -2$

$$a_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

So

$$\begin{aligned} a_0(x) &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \\ &= \frac{(x - (-1))(x-3)(x-6)}{(-3-(-1))(-3-3)(-3-6)} \\ &= \frac{(x+1)(x-3)(x-6)}{(-2)(-6)(-9)} \\ &= \tfrac{1}{108}(x+1)(x-3)(x-6) \end{aligned}$$

$$\begin{aligned} a_1(x) &= \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\ &= \frac{(x - (-3))(x-3)(x-6)}{(-1-(-3))(-1-3)(-1-6)} \\ &= \frac{(x+3)(x-3)(x-6)}{(2)(-4)(-7)} \\ &= \tfrac{1}{56}(x+3)(x-3)(x-6) \end{aligned}$$

$$\begin{aligned} a_2(x) &= \frac{(x - (-3))(x - 1(-1))(x-6)}{(3-(-3))(3-(-1))(3-6)} \\ &= \frac{(x+3)(x+1)(x-6)}{(6)(4)(-3)} \\ &= \tfrac{1}{72}(x+3)(x+1)(x-3) \end{aligned}$$

$$
\begin{aligned}
a_3(x) &= \frac{\left(x-(-3)\right)\left(x-(-1)\right)(x+3)}{\left(6-(-3)\right)\left(6-(-1)\right)(6-3)} \\
&= \frac{(x+3)(x+1)(x-3)}{(9)(7)(3)} \\
&= \tfrac{1}{189}(x+3)(x+1)(x-3)
\end{aligned}
$$

Then
$$
\begin{aligned}
P_3(x) &= \sum_{k=0}^{3} a_k y_k \\
&= a_0 y_0 + a_1 y_1 + a_2 y_2 + a_3 y_3 \\
&= 2\left(-\tfrac{1}{108}(x+1)(x-3)(x-6)\right) + 4\left(\tfrac{1}{56}(x+3)(x-3)(x-6)\right) \\
&\qquad +1\left(-\tfrac{1}{72}(x+3)(x+1)(x-6)\right) - 2\left(\tfrac{1}{189}(x+3)(x+1)(x-3)\right) \\
&= -\tfrac{1}{54}(x+3)(x-3)(x-6) + \tfrac{1}{14}(x+3)(x+1)(x-3) \\
&\qquad -\tfrac{1}{72}(x+3)(x+1)(x-6) - \tfrac{2}{189}(x+3)(x+1)(x-3)
\end{aligned}
$$

To verify that this polynomial does pass through the given points

When $x = -3$

$$
\begin{aligned}
P_3(-3) &= -\tfrac{1}{54}(-3+1)(-3-3)(-3-6) + 0 + 0 + 0 \\
&= -\tfrac{1}{54}(-2)(-6)(-9) \\
&= \frac{-108}{-54} \\
&= 2
\end{aligned}
$$

Hence $P_3(x)$ passes through $(-3, 2)$ as required. Similarly,

$$
\begin{aligned}
P_3(-1) &= 0 + \tfrac{1}{14}(-1+3)(-1-3)(-1-6) + 0 + 0 \\
&= \tfrac{1}{14}(+2)(-4)(-7) \\
&= 4
\end{aligned}
$$

$$
\begin{aligned}
P_3(3) &= -\tfrac{1}{72}(6)(4)(-3) = 1 \\
P_3(6) &= -\tfrac{2}{189}(9)(7)(3) = -2
\end{aligned}
$$

An alternative approach using Gaussian row reduction

Substitution into a formula is all very well – but where does the formula come from?

To find the interpolating polynomial $p(x)$ of degree $n,$ for $n+1$ data points, let
$p(x) = a_0 + a_1 x + ... + a_n x^n$

Then use the $n+1$ equations
$p(x_0) = f(x_0) = y_0$
$p(x_1) = f(x_1) = y_1$
...
$p(x_n) = f(x_n) = y_n$

to set up a system of linear simultaneous equations
$a_0 + a_1 x_0 + ... + a_n x_0^n = y_0$
$a_0 + a_1 x_1 + ... + a_n x_1^n = y_1$
...
$a_0 + a_1 x_n + ... + a_n x_n^n = y_n$
and solve them.

The solution is facilitated by writing the system in matrix form

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Substitution of the values $x_0,\ x,\ ...x_n, y_0,\ y_1\ ...y_n$ creates a matrix problem to be solved, by, for example, Gaussian elimination.

Example

Find the interpolating quadrant polynomial passing through the points (0,1) , (1,3) , (2,10). Use this to estimate $f\left(\frac{3}{2}\right)$

Solution

Let the interpolating polynomial be

$$p(x) = a_0 + a_1 x + a_2 x^2$$

Substitution into

$p(x_0) = a_0 + a_1 x_0 + a_2 x_0^2$
$p(x_1) = a_0 + a_1 x_1 + a_2 x_1^2$
$p(x_2) = a_0 + a_1 x_2 + a_2 x_2^2$

gives

$$p(0) = a_0 + 0a_1 + 0a_2 = 1$$
$$p(1) = a_0 + a_1 + a_2 = 3$$
$$p(2) = a_0 + 2a_1 + 4a_2 = 10$$

Hence the matrix representation

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 10 \end{bmatrix}$$

We use Gaussian elimination to solve this equation

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 4 & 10 \end{array}\right] = (2)-(1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 6 \end{array}\right] = (3)-2(2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{array}\right] = (3)-(1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{5}{2} \end{array}\right] = (3) \div 2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{2} \end{array}\right] (2)-(3)$$

Hence

$$\begin{aligned} a_0 &= 1 \\ a_1 &= -1/2 \\ a_2 &= 5/2 \\ \therefore\ p(x) &= 1 - \tfrac{1}{2}x + \tfrac{5}{2}x^2 \\ \therefore\ p\left(\tfrac{3}{2}\right) &= 1 - \tfrac{1}{2} - \tfrac{3}{2} + \tfrac{5}{2}\left(\tfrac{3}{2}\right)^2 \\ &= 1 - \tfrac{3}{4} + \tfrac{15}{8} \\ &= 1 - \tfrac{8\text{-}6\text{+}15}{8} = \tfrac{17}{8} \end{aligned}$$

Towards a proof of the interpolation polynomial

This process gives us a hint as to how we might go about proving the formula for Lagrange's interpolating polynomial. Let us start with the simple case first, where we want the liner interpolating polynomial.

The 2nd degree Lagrange interpolating polynomial is

$$p_2(x) = \sum_{k=0}^{1} a_k y_k = a_0 y_0 + a_1 y_1$$

where

$$a_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{1} \frac{x - x_i}{x_k - x_i}$$

That is

$$a_0 = \prod_{\substack{i=0 \\ i \neq k}}^{1} \frac{x - x_i}{x_k - x_i} = \frac{x - x_1}{x_0 - x_1} \qquad a_1 = \prod_{\substack{i=0 \\ i \neq k}}^{1} \frac{x - x_i}{x_k - x_i} = \frac{x - x_0}{x_1 - x_0}$$

So we are seeking to prove that

$$p_2(x) = a_0 y_0 + a_1 y_1 = a_0\left(\frac{x - x_1}{x_0 - x_1}\right) + a_1\left(\frac{x - x_0}{x_1 - x_0}\right)$$

To show this let the polynomial take the form

$$p(x) = a_0 + a_1 x$$

Substituting the pairs of values (x_0, y_0) and (x_1, y_1) we get the pair of simultaneous equations

$$a_0 + a_1 x_0 = y_0$$
$$a_0 + a_1 x_1 = y_1$$

...

which we have to solve.

Writing this in matrix form and using Gaussian row reduction

$$\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & x_0 & y_0 \\ 0 & x_1 - x_0 & y_1 - y_0 \end{array}\right] \quad (1)-(2)$$

$$\left[\begin{array}{cc|c} 1 & x_0 & y_0 \\ 0 & 1 & \dfrac{y_1 - y_0}{x_1 - x_0} \end{array}\right] \quad (2)\div(x_1 - x_0)$$

$$\left[\begin{array}{cc|c} 1 & x_0 & y_0 \\ 0 & x_0 & x_0\left(\dfrac{y_1 - y_0}{x_1 - x_0}\right) \end{array}\right] \quad (2)\times x_0$$

Hence

$$a_0 = y_0 - x_0\left(\frac{y_1 - y_0}{x_1 - x_0}\right)$$

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0}$$

Substituting into
]

$$p(x) = a_0 + a_1 x$$

$$p(x) = y_0 - x_0\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + a_1\left(\frac{y_1 - y_0}{x_1 - x_0}\right)x$$

Placing everything over a common denominator and collecting terms

$$\begin{aligned} p(x) &= y_0\left(\frac{x_1 - x_0}{x_1 - x_0}\right) - x_0\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + a_1\left(\frac{y_1 - y_0}{x_1 - x_0}\right)x \\ &= \frac{y_0\left(x_1 - x_0 + x_0 + x\right) + y_1\left(x - x_1\right)}{x_1 - x_0} \\ &= a_0\left(\frac{x - x_1}{x_0 - x_1}\right) + a_1\left(\frac{x - x_0}{x_1 - x_0}\right) \end{aligned}$$

This is what we were required to prove.

To prove the result in general for an interpolating polynomial of any degree, *n*, we would need to iterate this process and use a mathematical induction. This could involve a good deal of nasty algebra!