Linear dependence and independence - singular and non-singular matrices

Prerequisites

You should be familiar with the addition and multiplication of matrices and be able to find the determinant of 2×2 and 3×3 matrices, as well as the inverse of a 2×2 matrix.

Matrices that do not have an inverse: singular matrices

Compare the following 2×2 matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

We shall show that the second matrix has an inverse, but the first does not. For the second matrix its inverse is

$$B^{-1} = \frac{1}{5-4} \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

But when we attempt to find the inverse of A we encounter a difficulty.

$$A^{-1} = \frac{1}{4\text{-}4} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \frac{1}{0} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

But we cannot divide by 0; hence this operation for this matrix is ill-defined and A cannot have an inverse. This failure of A to have an inverse is linked to the value of its determinant. The determinant of a matrix is

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

In the case of $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, A does not have an inverse because det $A = 0$. Matrices for which the determinant is zero are called *singular* matrices. When $\det A \neq 0$ the matrix is called *non-singular.*

> The general rules is that for any square $n \times n$ matrix
> $\det A = 0 \Leftrightarrow A$ is a singular matrix $\Leftrightarrow$ A does not have an inverse

(In this statement the symbol $\Leftrightarrow$ means, for example, "$\det A = 0$ *implies* A is singular and A is singular *implies* $\det A = 0$". This symbol is called the *biconditional*."

Example (1)

Let

$$A = \begin{pmatrix} 8 & -4 \\ 6 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & -2 \\ -1 & -2 \end{pmatrix}$$

One of these has an inverse and the other does not. Find the inverse of the one that does have one, and explain why the other lacks an inverse.

Solution

$$\det A = \begin{vmatrix} 8 & -4 \\ 6 & -3 \end{vmatrix} = 8 \times -3 - (-4 \times 6) = -24 + 24 = 0$$

$$\det B = \begin{vmatrix} 4 & -2 \\ -1 & -2 \end{vmatrix} = 4 \times -2 - (-2 \times -1) = -8 - 2 = -10$$

So A is a singular matrix, and therefore does not have an inverse. The inverse of B is

$$B^{-1} = \frac{1}{\det B}\begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2 & -2 \\ -1 & -4 \end{pmatrix}$$

Linear dependence and independence

There is further feature of A that is linked to the reason why A does not have an inverse. Take another look at $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Notice that the second row is twice the first row. We say that the second row is a *linear combination* of the first because it can be obtained from the first by just multiplying it. For matrices of any size, when we multiply and add rows (or columns) of a matrix, then we are forming what is called a *linear combination* of them. When one row is a multiple of another or the sum of multiples of other rows, then we say that the rows are linearly dependent. All the rows are linearly dependent if one of them is the linear combination of any of the others. If a matrix is not linearly dependent then it is said to be *linearly independent.*

Example (2)

Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Given that one of these is linearly dependent the other is linearly independent find which is which.

Solution

A matrix is linearly dependent if one row is a linear combination of the others. In *A*, the third row is the sum of the first and second rows.

$\begin{pmatrix}1 & 0 & -1\end{pmatrix} = \begin{pmatrix}1 & -1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1 & -1\end{pmatrix}$

So it is *A* that is linearly dependent and *B* must be linearly independent, since we are told that only one of these is linear dependent.

$$A = \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{pmatrix} \qquad B = \begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$$

If the rows are linearly dependent, then the columns are also linearly dependent.

Example (3)

For $A = \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{pmatrix}$ find a linear combination of the first two columns that gives the third column.

Solution

To solve this we are being asked to find numbers *a* and *b* such that

$$a\begin{pmatrix}1\\ 0\\ 1\end{pmatrix} + b\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix} = \begin{pmatrix}0\\ -1\\ -1\end{pmatrix}$$

This leads to the equations

$a - b = 0$

$b = -1$

$a = -1$

These are consistent, and the solution is $a = -1, b = -1$.

There is a relationship between whether a matrix is singular and whether it is linearly dependent.

Example (4)

Let $A = \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{pmatrix} \qquad B = \begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$

Find the determinant of each.

Solution

$$\det A = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} - 1\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 + 1 + 0 = 0$$

$$\det B = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 + 0 + 1 = 2$$

The solution to this problem invites the conjecture that

A is linearly dependent $\Leftrightarrow$ A is singluar$(\det A = 0)$
A is linearly independent $\Leftrightarrow$ A is non-singluar$(\det A \neq 0)$

This conjecture is *true*. Let us prove this for 2×2 matrices.

Example (5)

Prove the above conjecture for any 2×2 matrix, *A*.

Solution

First let us prove

A is linearly dependent $\Rightarrow$ A is singluar$(\det A = 0)$

What *A* is linearly dependent says, is that one column of *A* is a linear combination of the other. Let

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

Then the only way the second column can be a linear combination of the first is if it is a multiple of it. That is

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = k\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$A = \begin{pmatrix} a_1 & ka_1 \\ a_2 & ka_2 \end{pmatrix}$$

and

$$\det A = \begin{vmatrix} a_1 & ka_1 \\ a_2 & ka_2 \end{vmatrix} = ka_1a_2 - ka_1a_2 = 0$$

Now we look at the converse statement

A is singluar$(\det A = 0) \Rightarrow A$ is linearly dependent

If *A* is singular then

$$\det A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = 0$$

$$a_1b_2 = a_2b_1$$

$$b_2 = \frac{a_2}{a_1}b_1$$

$$\frac{b_1}{a_1}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ \frac{a_2}{a_1}b_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

So the second column is a multiple of the first with scalar b_1/a_1.

Since matrices are either singular or non-singular, or either dependent or independent, it follows that the statement

A is linearly dependent $\Leftrightarrow$ A is singluar$(\det A = 0)$

is equivalent to the statement

A is linearly independent $\Leftrightarrow$ A is non-singluar$(\det A \neq 0)$

This proves the conjecture for any 2×2 matrix; however, the conjecture is true for any square matrix, $n \times n$, regardless of the size of *n*.

Example (6)

$$P = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & -1 \\ -1 & -6 & 5 \end{pmatrix}$$

(a) Show that P is singluar

(b) Find a linear combination of the first two columns that gives the third column.

Solution

(a) $$\det \text{P} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -1 \\ -1 & -6 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & -1 \\ -6 & 5 \end{vmatrix} - 1\begin{vmatrix} -1 & 3 \\ 5 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ -1 & -6 \end{vmatrix}$$

$$= 1(14) \ -1(-14) + 2(-14)$$

$$= 14 + 14 - 28$$

$$= 0$$

(b) Let

$$\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \alpha\begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Uncoupling means to unite each row of this vector equation separately. Hence

(1) $1 = -\alpha + 2\beta$

(2) $2 = 4\alpha - \beta$

(3) $3 = -6\alpha + 5\beta$

These are three simultaneous equations in two unknowns. If *P* were not linearly dependent and having zero determinant, then this system would not have a unique solution. However, since det *P* = 0, we know it does.

(1) × 4 $\quad 4 = -4\alpha + 8\beta \quad$ (4)

(2) + (4) $\quad 7 = 7\beta$

$\beta = 7\beta$

$\beta = 1 \qquad \alpha = 1$

Hence

$$\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

For a linearly independent (non-singular) matrix we cannot obtain a solution to the attempt to unite one column (or row) as a linear combination of the others.

Example (7)

Let $Q = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & -1 \\ -1 & -6 & 4 \end{pmatrix}$

(*a*) Show that *Q* is linearly independent

(*b*) Prove that the attempt to write the first column as a linear combination of the others leads to a contradiction.

Solution

(*a*) $\det Q = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -1 \\ -1 & -6 & 4 \end{vmatrix}$

$$= 1\begin{vmatrix} 1 & -4 \\ -6 & 4 \end{vmatrix} - 1\begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ -1 & -6 \end{vmatrix}$$

$= 1(8) - 1(-13) + 2(-14)$

$= 8 + 13 - 28$

$= -7$

$\neq 0$

Hence $\det Q \neq 0$ and *Q* is non-singular.

(b) Let $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ 70

(1) $1 = -\alpha + 2\beta$

(2) $3 = 4\alpha - \beta$

(3) $-1 = -6\alpha + 4\beta$

Solving equations (1) & (2) gives $\alpha = 1, \beta = 1$ but when we substitute these values into (3) we obtain

$-1 = -6 + 4$
$-1 = -2$

which is a contradiction. Since this is a contradiction it follows that the first column cannot be a linear combination of the other two.

Definition of linear dependence for vectors in general.

Linear dependence

The vector $\underline{\mathbf{a}}$ is said to be linearly dependent on the vectors $\underline{\mathbf{b_1}}, \underline{\mathbf{b_2}}, \underline{\mathbf{b_3}} \ldots \underline{\mathbf{b_n}}$

if and only, there exists numbers $\beta_1, \beta_2, \beta_3 \ldots \beta_n$

$\underline{\mathbf{a}} = \beta_1 \underline{\mathbf{b}_1} + \beta_2 \underline{\mathbf{b}_2} + \ldots + \beta_n \underline{\mathbf{b}_n}$

The numbers could be of any type – real or complex.

Summary

We now summarise our findings.

For a square $n \times n$ matrix A

$\det A = 0$
A is singular
A^{-1} does not exist
A is linearly dependent
$\underline{\mathbf{a}} = \beta_1 \underline{\mathbf{b}_1} + \beta_2 \underline{\mathbf{b}_2} + \ldots + \beta_n \underline{\mathbf{b}_n}$ can be solved for any rows or columns

} All these statements are equivalent

The last condition means that there does exist a solution to the system of simultaneous equations formed when we attempt to form one column or row of A as a linear combination of the others. Conversely,

$\det A \neq 0$
A is non-singular
A^{-1} exists
A is linearly independent
$\underline{\mathbf{a}} = \beta_1 \underline{\mathbf{b}_1} + \beta_2 \underline{\mathbf{b}_2} + ... + \beta_n \underline{\mathbf{b}_n}$ cannot be solved for any rows or columns

} All these statements are equivalent