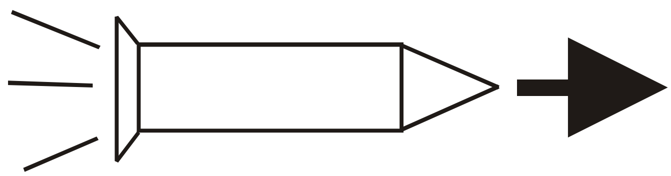
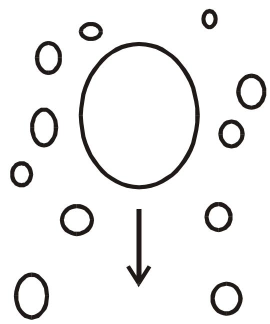
Linear Motion of a Body of Variable Mass

Rockets and raindrops

Up to this stage we have only considered the laws governing the motion of particles of constant mass. But in many cases the mass of a particle will change as it moves. The two standard examples are rockets and raindrops.

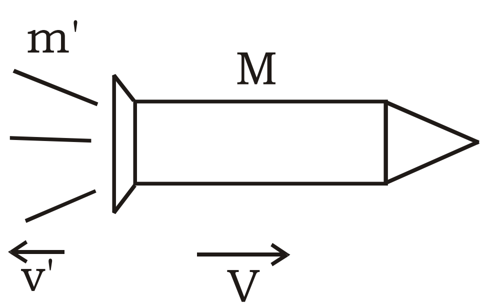


As the rocket moves forward it expels gas. The loss of gas makes the rocket lighter.



As the raindrop falls through a vapour cloud more of the vapour adheres to it – it gains mass.

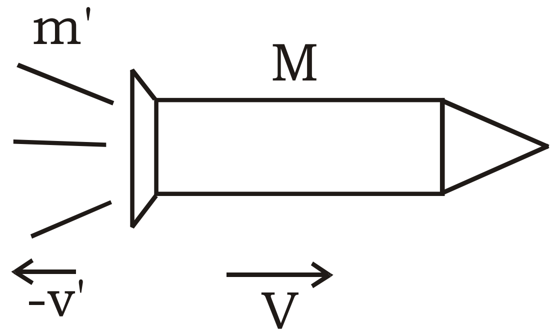
The motion of the a missile is an instance of Newton's Third Law – the missile travels forward because vapour is projected from its rear end. The missile and the vapour are recoiling from each other – total momentum is conserved.



A missile of mass M moves with speed V, ejecting gas of mass m' at speed v'. Conservation of momentum tells us that:

Mv = m'v'

Here we are using only magnitudes, but strictly the velocities of the rocket and the expelled gases are vectors and should be measured according to a sign convention – for example, that movement to the right is positive, to the left negative. Hence

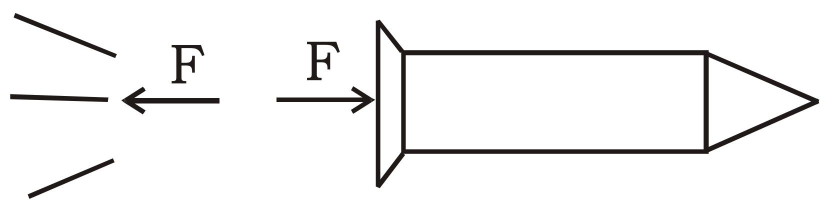


Then total momentum is conserved

$$MV + m'v' = 0$$

However, since the mass of the rocket is constantly changing and the gas is ejected continuously, this relationship is not very useful in solving such questions as "how fast is the rocket travelling at a given instance?"

At any given instance the rocket exerts a force, F, on the ejected gas. Likewise, in accordance with Newton's Third Law, the gas exerts a force of equal magnitude on the rocket.



The force acting on the rocket will cause it to accelerate. The magnitude and direction of this force is given by the "full" version of Newton's Second Law. Up to this time you have probably encountered Newton's 2nd Law in the "restricted" version. That is, as:

$$F = ma$$

Force = mass × acceleration.

This version applies only in contexts where the mass of the object is constant. The "full" version is required when we consider linear momentum of a body of variable mass.

Newton's Second Law

When an external force is applied to a body the rate of increase of momentum produced is directly proportional to the applied force.

An instantaneous rate is given by the first derivative; thus this law is written as

$$\frac{d}{dt}(mv) \propto F$$

The S.I. units of mass are kilograms (kg), of velocity metres/second (ms^{-1}), and when mass and velocity are measured in these units the constant of proportionality is 1 and Newton's Second Law is:

$$\frac{d}{dt}(mv) = F$$

Here $m = m(t)$ is the mass of the object at time t and $v = v(t)$ is the velocity of the object at time t.

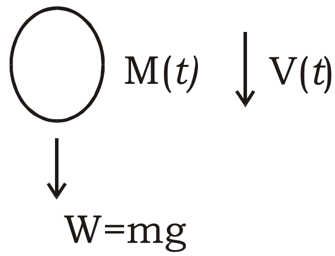
It is often possible to integrate this law directly to obtain an expression for $v = v(t)$, the velocity of the particle at time t. This is the case in the following example.

Example (1)

The initial mass of a raindrop is M_0. It falls from an initial velocity V_0 through a cloud of still water vapour accumulating water from the cloud as it does so. The mass of the drop at time t is M and its velocity is V. It is assumed that the mass of the drop is governed by the equation:

$$M = M_0 e^{\lambda t} \qquad \lambda > 0$$

(i) Ignoring air-resistance, find an expression for V.
(ii) Sketch the graph of V against t.



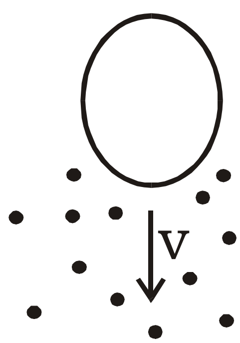
Newton's Second Law gives:-

$$F = \frac{d(mv)}{dt}$$
$$\therefore M_0 e^{\lambda t} g = \frac{d}{dt}\left(M_o e^{\lambda t} v\right)$$
$$\therefore e^{\lambda t} v = \frac{g e^{\lambda t}}{\lambda} + c$$

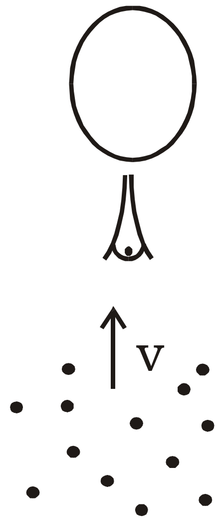
Thus, we can solve problems by integrating Newton's 2nd law in the form:

$$F = \frac{d}{dt}(mv)$$

However, we must issue a word of warning. This would only be applicable when the object by accumulating mass does not accumulate mass that imparts an impulse to it. This means that the mass increment must be at rest, from an external viewpoint so its momentum is zero. If the mass increment is at rest from an external viewpoint from the viewpoint of the object the mass increment is travelling toward the object with magnitude *v* and in the opposite direction.



From an external viewpoint the raindrop is travelling towards the water vapour with velocity *v*. The water vapour is still and has no momentum.



From the viewpoint of the raindrop the water vapour is travelling towards it with speed *v*.

If the mass increment is not still when viewed from an external frame of reference then when the particle accumulates it there will be an impulse imparted. The impulse will alter the force acting on the particle and it will no longer be true that

$$F = \frac{d}{dt}(mv)$$

because F will not take into account the effect of this impulse.

Thus we can only integrate

$$F = \frac{d}{dt}(mv)$$

directly when the mass increment has no momentum. Often this is assumed in questions but this would not apply in cases where a rocket, for instance, moves by propelling gas in the form of burnt fuel.

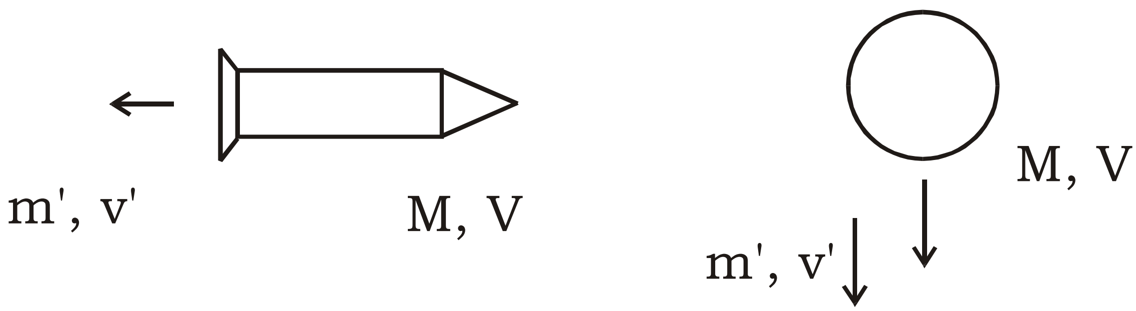
Thus Newton's Second Law is

$$\frac{d}{dt}(mv) = F$$

Likewise, it is tempting to differentiate this result to obtain:

$$F = m\frac{dv}{dt} + v\frac{dm}{dt}$$

However, this would be a mistake and only applies when the mass increment has no momentum. Hence, either the mass of the object is not constant or the velocity of the mass increment is 0. If the mass were constant *dm/dt* would equal 0 and the result would reduce to the familiar form of Newton's Second Law: F = ma. We are dealing with precisely the case where the mass is not constant and $dm/dt \neq 0$, and we now consider the case where the velocity of the mass increment is not 0. The mass cannot be changing if it is not either gaining or losing mass – let the mass that is either joining or is being ejected have a velocity v' at the moment of impact or separation.



The rocket ejects mass m' with velocity v'; the raindrop accumulates mass m' with velocity v'.

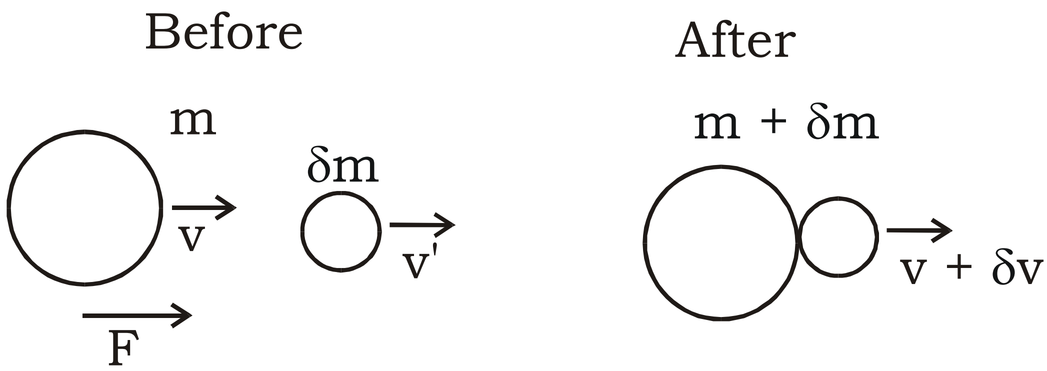
A body gains mass by joining with particles that have a velocity of their own. A body loses mass by ejecting particles that likewise have a velocity of their own.

We must, therefore, derive a "correct" formula from first principals.

There are, in fact, two cases to consider – firstly, where the object gains mass as in the rain drop, secondly, where the object loses mass, as in the rocket.

First Case (mass increment)

An object of mass M is travelling with velocity V and is subject to an external force F.
In time ∂t it is joined by a mass ∂m travelling with velocity v'.



Let $m = m(t)$ be the mass at time t.
Let $v = v(t)$ be the velocity at time t.
Let δm be the mass increment after time δt.
Let v' be the velocity of the mass increment.
Then $m + \delta m$ is the mass of the coalesced particle at $t + \delta t$.
Also $v + \delta v$ is the velocity of the coalesced particle at $t + \delta t$.

Then:

increase in momentum = momentum after – momentum before

$$\begin{aligned}
\delta p &= (m + \delta m)(v + \delta) - mv - \delta m v' \\
&= mv + m\delta v + v\delta m + \delta m \delta v - mv - \delta m v' \\
&= m\delta v + v\delta m - \delta m v' + \delta m \delta v \\
&= m\delta v + (v - v')\delta m + \delta m \delta v
\end{aligned}$$

Therefore the rate of change of momentum is approximately given by:-

$$\frac{\delta p}{\delta t} \approx \frac{1}{dt}\left(m\delta v + (v - v')\delta m + \delta m \delta v\right)$$

$$\frac{\delta p}{\delta t} \approx m\frac{\delta v}{\delta t} + (v - v')\frac{\partial m}{\partial t} + \frac{\partial m \partial v}{\partial t}$$

Newton's Second Law gives:

$$F = \lim_{\partial t \to 0}\left(\frac{\delta p}{\delta t}\right)$$

$$= \lim_{\delta t \to 0}\left(m\frac{\delta v}{\delta t} + (v - v')\frac{\delta m}{\delta t} + \frac{\delta m \delta v}{\delta t}\right)$$

$$= m\frac{dv}{dt} + (v - v')\frac{dm}{dt}$$

Since $\lim_{\partial t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}$, $\lim_{\partial t \to 0} \frac{\delta m}{\delta t} = \frac{dm}{dt}$ and $\lim_{\partial t \to 0} \frac{\delta m \delta v}{\delta t} = 0$

Hence the form of the differential equation governing linear motion of a body undergoing an increment of mass is:

$$F = m\frac{dv}{dt} + (v - v')\frac{dm}{dt}$$

The expression v - v' represents the velocity with which the object is gaining on the mass increment. Setting:

$$u = |v - v'|$$

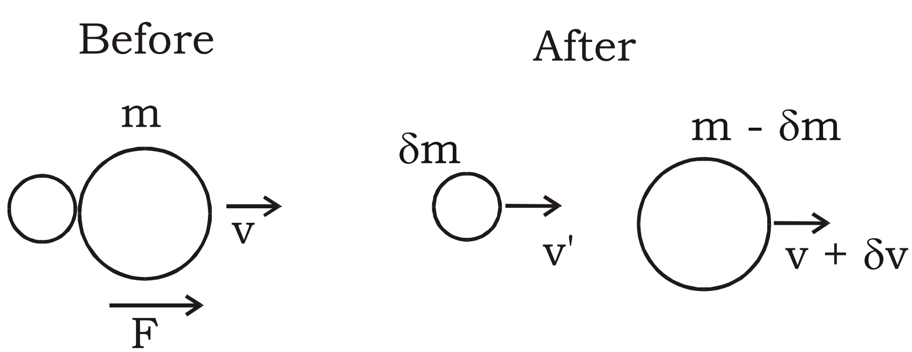
We can write:

$$F = m\frac{dv}{dt} + u\frac{dm}{dt}$$

where *u* is the speed of the object relative to the mass increment – or, which is now the same thing, the speed of the mass increment relative to the mass. If you could imagine yourself traveling with the object, the mass increment would appear to be traveling with speed *u* towards you.

<u>Second Case (mass decrement)</u>

An object of mass m is traveling with velocity v and is subject to an external force *F*. In time δt it loses a mass δm traveling with velocity v'.



Let $m = m(t)$ be the mass at time t.
Let $v = v(t)$ be the velocity at time t.
Let δm be the mass decrement after time t.
Let v' be the velocity of the mass decrement.
Then $m - \delta m$ is the mass of the object at $t + \delta t$.
And $v - \delta v$ is the velocity of the object at $t + \delta t$.

Then:

increase in momentum = momentum after – momentum before.

Here momentum after $= (m - \delta m)(v + \delta v) + \delta m v'$
momentum $= mv$

Hence:

$$\delta p = (m - \delta m)(v + \delta v) + \delta m v' - mv$$
$$= mv + m\delta v - \delta m v - \delta m \delta v + \delta m v' - mv$$
$$= m\delta v - v\delta m + v'\delta m - \delta m \delta v$$
$$\therefore \frac{\delta p}{\delta t} \approx m\frac{\delta v}{\delta t} + (v' - v)\frac{\delta m}{\delta t} - \frac{\delta m \delta v}{\delta t}$$

Hence, applying Newton's 2nd Law:

$$F = \lim_{\delta t \to 0}\left(m\frac{\delta v}{\delta t} + (v' - v)\frac{\delta m}{\delta t} - \frac{\delta m \delta v}{\delta t}\right)$$
$$= m\frac{dv}{dt} + (v' - v)\frac{dm}{dt}$$

Let us compare the governing differential equations in the two cases:

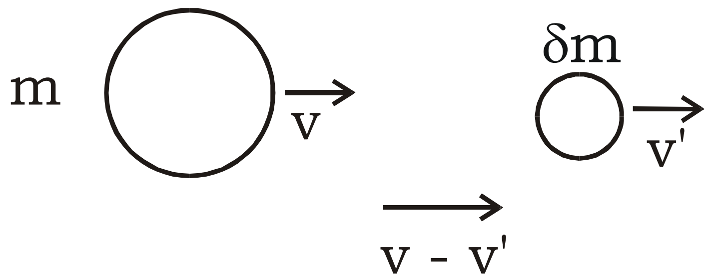
Mass increment:
$$F = m\frac{dv}{dt} + (v - v')\frac{dm}{dt}$$

Mass decrement:
$$F = m\frac{dv}{dt} + (v' - v)\frac{dm}{dt}$$

We can see that the two terms differ only in the expression $v - v'$ (mass increment) or $v' - v$ (mass decrement). Note $v' - v = -(v - v')$.

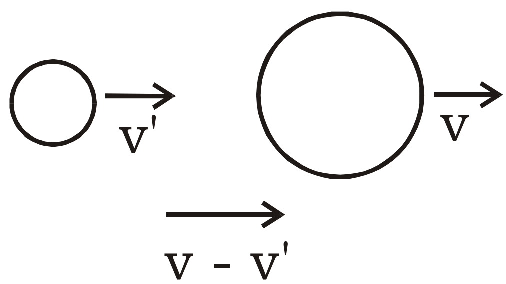
$v - v'$ is the velocity of the object relative to the particle.

Mass increment:



The object appears to be gaining on the mass increment.

Mass decrement:



The object appears to be moving away from the mass decrement.

We set $u = | v - v' |$ and write the equation for both cases – increment and decrement – as:-

$$F = m\frac{dv}{dt} + u\frac{dm}{dt}$$

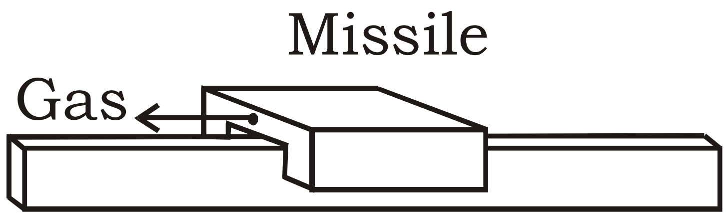
Then u is the speed of the mass increment or decrement relative to the object. It is no longer a vector and it is always understood to be a positive quantity. It is understood to be the speed with which coalescing particles travel towards the object in the case of mass increment, or the speed with which ejected particles travel away from an object in the case of mass decrement.

A further note. The term F refers to external forces acting on the system. If, for example, a rocket is running on a frictionless track, then there are no external forces and $F = 0$. This is the case in our first example.

We now proceed to illustrate the application of this equation to specific examples.

Example (2)

A missile, initially of total mass M_0 *kg*, is launched horizontally from rest along a smooth, frictionless track.



Gas is ejected from the missile at a rate of *k kg* per second and with a speed $u\ ms^{-1}$ relative to the missile. If the mass of the missile is *m* at a time *t* and its velocity is *v* at time *t*, find *v* in terms of M_0, *k*, *u* and *t*. Given $u = 20ms^{-1}$ the particle ceases to eject gas when its mass is $\frac{1}{2}M_0$. What is its maximum speed? The mass ejected in t seconds = *kt.*

$\therefore M = M_0 - kt.$

Newton's Second Law gives:

$$F = m\frac{dv}{dt} + u\frac{dm}{dt}$$

Here, there are no external forces, since the track is smooth, hence $F = 0$; substituting $m = M_0 - kt$

$$(M_0 - kt)\frac{dv}{dt} + u\frac{d}{dt}(M_0 - kt) = 0$$

$$\therefore (M_0 - kt)\frac{dv}{dt} - ku = 0$$

$$\therefore \frac{dv}{dt} = \frac{ku}{M_0 - kt}$$

$$\therefore v = \int \frac{ku}{M_o - kt} \cdot dt$$

$$= -u\ln(M_0 - kt) + c$$

When $t = 0$, $v = 0$, hence c = $u \ln M_0$

$$\therefore v(t) = u\ln(M_0) - u\ln(M_0 - kt)$$

$$= u\ln\left(\frac{M_0}{M_0 - kt}\right)$$

When $M_0 - kt = \frac{1}{2}M_0$ and $u = 20$

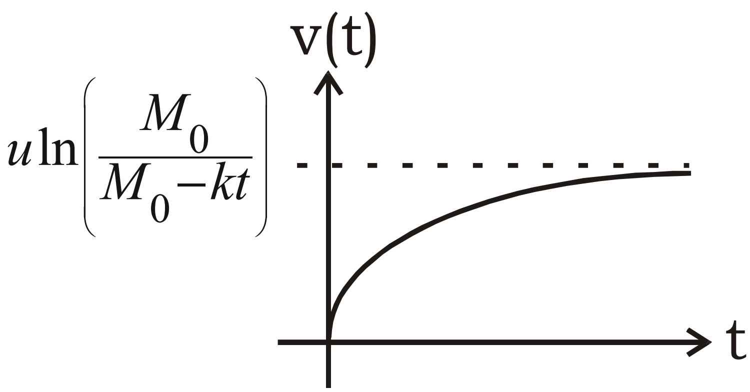
$$v(t) = 20 \cdot \ln\left(\frac{M_0}{\frac{1}{2}M_0}\right)$$

$$= 20.\ln\left(\frac{M_0}{\frac{1}{2}M_0}\right)$$

$$= 20.\ln 2$$

$$= 13.86$$

$$= 14\ ms^{-1}(2\ S.F.)$$



You may be asked to derive the equation of motion from first principals. Another example further illustrates this.

Example (3)

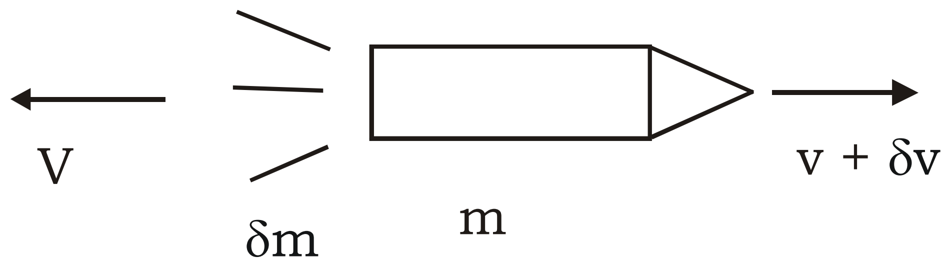
A rocket ejects fuel at a rate λ. The fuel is ejected backwards from the rocket with a constant velocity V relative to the rocket. Use conservation of linear momentum to show that

$$m\frac{dv}{dt} = \lambda V$$

The fuel is being used at the rate

$$\lambda = \frac{dm}{dt}$$

To solve this problem picture the rocket after it has ejected fuel, δm, at a speed V relative to the rocket.



increase in momentum backwards = increase in momentum forwards

$$\delta m V = m \delta v$$

$$\therefore \frac{\delta m}{\delta t} V = m \frac{\delta v}{\delta t}$$

And in the limit, when $\delta t \to 0$

$$\frac{dm}{dt} V = m \frac{dv}{dt}$$

i.e. $m\frac{dv}{dt} = \lambda V$

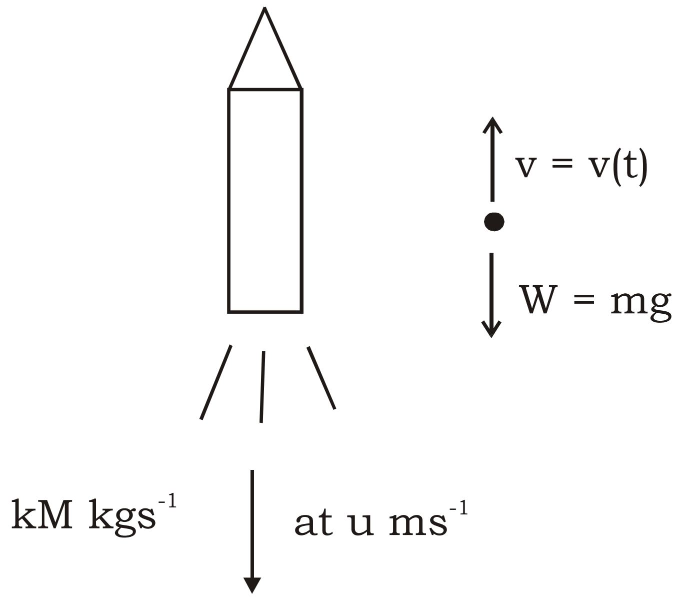
We now illustrate the case where external forces apply. In this third example a rocket is moving under gravity.

<u>Example (4)</u>

A rocket with initial mass M_0 is launched from rest. Burnt fuel is ejected downwards at a constant rate $M_0 k\ kg\ s^{-1}$. The velocity of the fuel relative to the rocket is $u\ ms^{-1}$. Ignoring air resistance (i) show that

$$(1 - kt)\frac{dv}{dt} = ku - g(1 - kt)$$

and (ii) demonstrate that the rocket cannot be launched unless $ku > g$.
(iii) find v in terms of t.



At time $t = 0$
$M(t) = M_0$
$v(t) = 0$

At time $t = 0$
$M(t) = M_0$
$v(t) = 0$

The rocket is moving under gravity. Since we are ignoring air-resistance the weight of the rocket is the only external force acting.

The mass of the rocket at time t is given by

$$m(t) = M_0 - kM_0 t = M_0(1 - kt)$$

Newton's Second Law states.

$$F = m\frac{dv}{dt} + u\frac{dm}{dt}$$
$$\therefore -m(t)g = m(t)\frac{dv}{dt} + u\frac{d}{dt}m(t)$$

$$\therefore -M_0\left(1-kt\right)g = M_0\left(1-kt\right)\frac{dv}{dt} + u\frac{d}{dt}M_0\left(1-kt\right)$$

$$\therefore -\left(1-kt\right)g = \left(1-kt\right)\frac{dv}{dt} - uk$$

$$\therefore \left(1-kt\right)\frac{dv}{dt} = uk - \left(1-kt\right)g$$

Shown.

(ii) To start moving

$$\frac{dv}{dt} > 0 \text{ when } t = 0$$

$$\therefore uk - \left(1-kt\right)g > 0 \text{ when } t = 0$$

$$\therefore uk - g > 0$$

$$\therefore ku > g$$

Shown.

(iii) We have $v = 0$ when $t = 0$. The equation is:

$$\left(1-kt\right)\frac{dv}{dt} = ku - g\left(1-kt\right)$$

$$\int dv = \int \frac{ku - g\left(1-kt\right)}{1-kt} \cdot dt$$

$$= \int\left(\frac{ku}{1-kt} + g\right)dt$$

$$\therefore v = -u\ln\left(1-kt\right) + gt + c$$

Substituting $v = 0$, $t = 0$, we obtain $c = 0$.

$$\therefore v = -u\ln\left(1-kt\right) + gt$$