Linear, non-linear relations and experimental laws

Linear proportionality and experimental laws

A law in experimental science is often expressed as an equation connecting physical quantities. For example, when an electrical current passes through a resistor, the loss of potential across the resistor is given by Ohm's law.

$V = IR$

Here V is the potential difference across a resistor, I is the current passing through the resistor, and R is the constant of proportionality – the resistance of the resistor.

We will continue with this example for the present. It is not really necessary to understand anything about electricity in order to understand the mathematical relationship between the quantities V and I. All that is needed is to understand that as a quantity, I, changes, so the quantity V is also affected, according to the rule (or law)

$V = IR$

This equation is a direct application of the topic of proportionality, and the student should be familiar with this topic as well as the topic of the equation of the straight line before commencing this chapter As this chapter also explores relations that are called "log/linear relations", the student should be familiar with logarithms and the exponential function. In addition, familiarity with basic curve sketching is also important.

But to return to the example, how is an equation of the form

$V = IR$

discovered? Essentially, this is through the collection of data from an experiment designed to test a hypothesis.

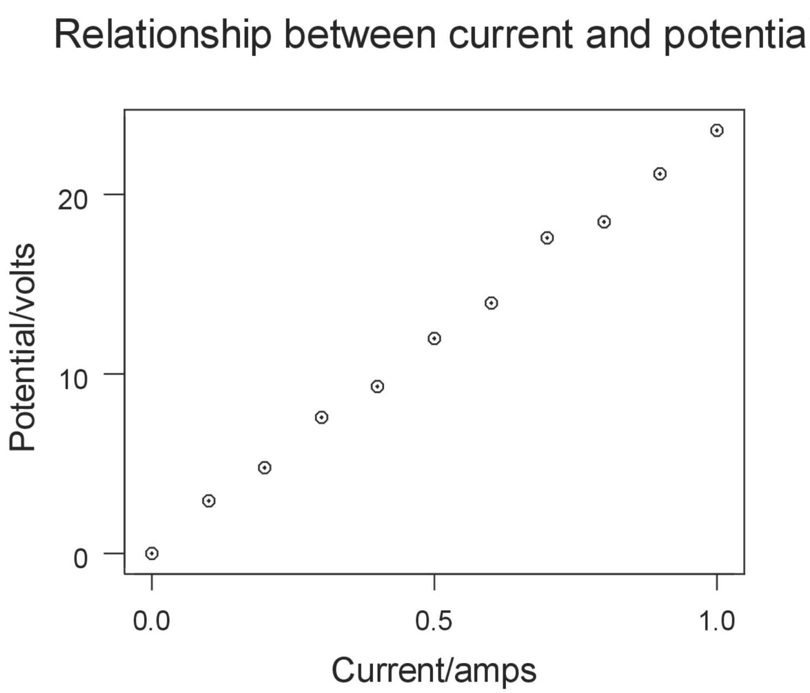
For example, suppose that our hypothesis is that potential difference is proportional to current across a resistor. That is, the hypothesis is

$V \propto R$

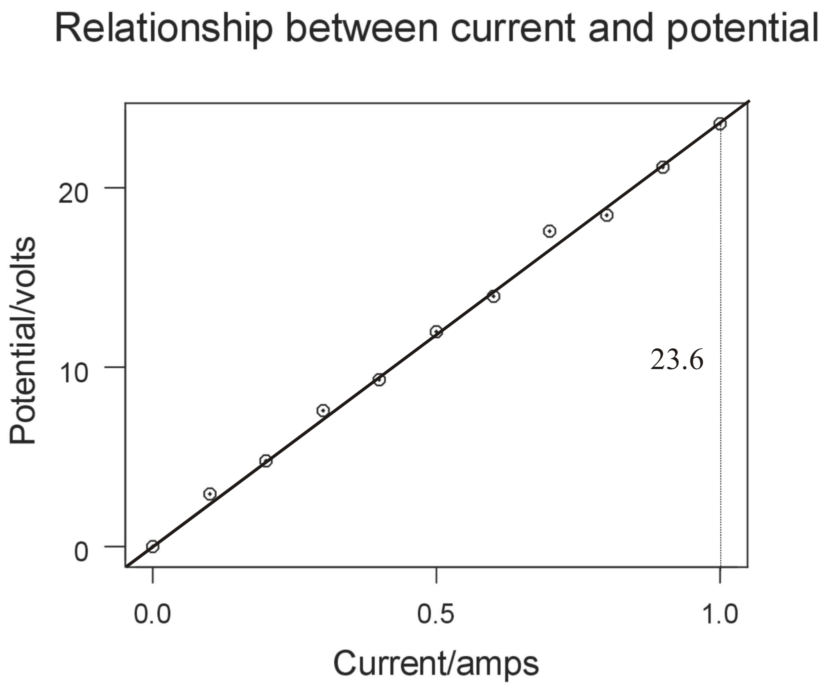
Let us imagine that in an experiment the following data was collected.

I / amps	V / volts
0.0	0.0
0.1	2.9
0.2	4.8
0.3	7.6
0.4	9.3
0.5	12.0
0.6	14.0
0.7	17.6
0.8	18.5
0.9	21.2
1.0	23.6

The next stage would be to plot a graph of the data.



We now fit a line to the graph. At this stage the line will be fitted "by eye". In a later chapter you can learn how use a mathematical technique of linear regression to find the equation of the straight line that comes "closest to the mostest" in some sense of what being close to a line might mean. Here we use our own judgement to find the line.



The hypothesis is

$V \propto R$

This translates into the linear relationship

$V = IR$

where R is the constant of proportionality. From the graph we can calculate R as

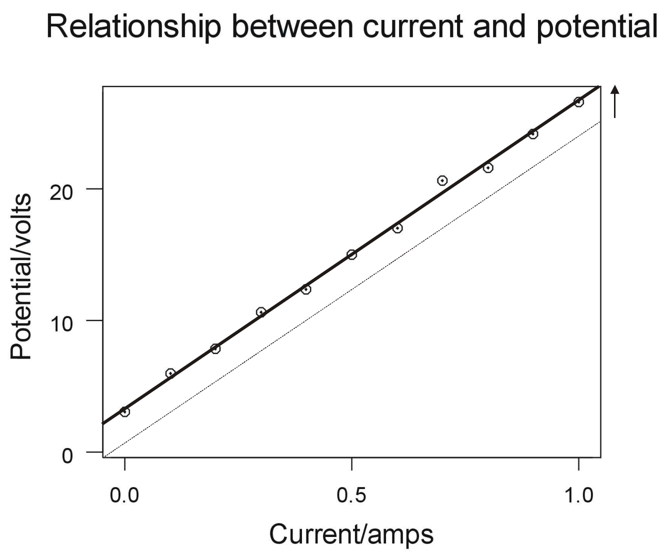
$$R = \frac{\Delta V}{\Delta I} = \frac{\text{change in } V}{\text{change in } I} = \frac{23.6}{1.0} = 23.6\Omega$$

(The units of resistance are ohms, symbol Ω.)

Sometimes an experimental relationship takes the form

$y = mx + c$

where c is the intercept on the vertical axis. This relationship can also occur where there is a systematic error, leading to the whole relationship being offset in some way. For instance, if the voltmeter used in the above experiment had not been properly zoered at the beginning of the experiment, all the data points could be systematically "out" by a constant amount, causing the graph to be shifted up or down.



Non-linear relationships and their reduction to linear forms

Often experimental laws involve non-linear relationships, such as equations of the form

$$y = ax^n + c$$

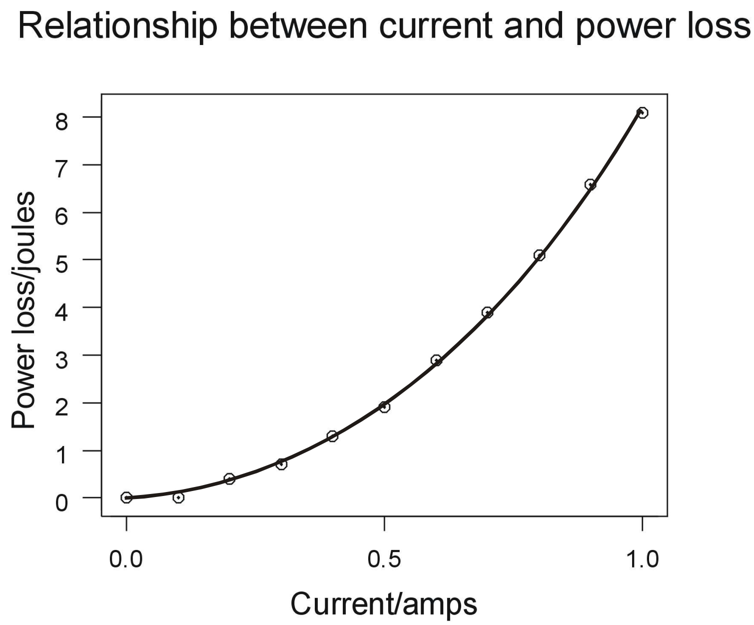
For example, power loss across a resistor is given by the equation

$$P = RI^2$$

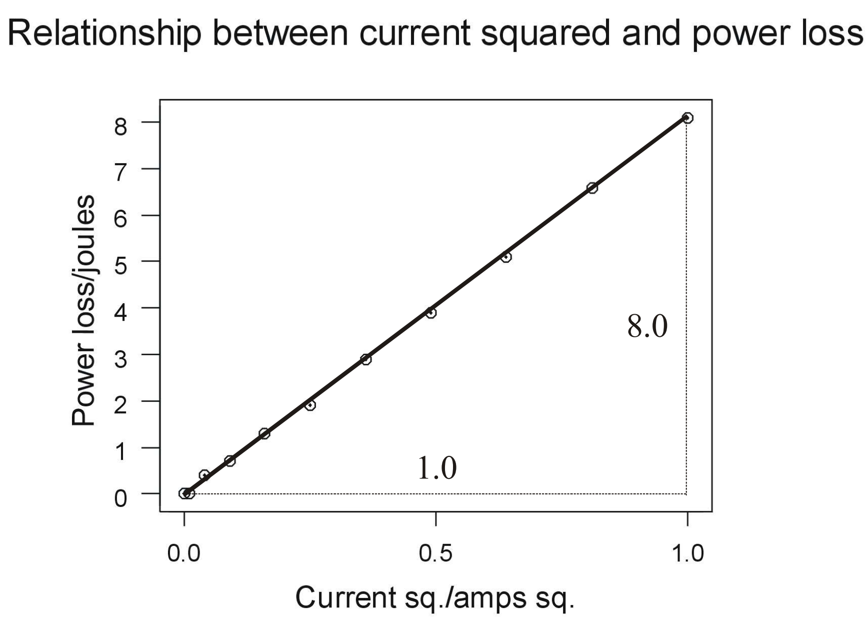
That is, the hypothesis is that power loss is proportional to the square of the current passing through a wire or resistor.

I / amps	I^2 / amps^2	P / joules
0.0	0.0	0.0
0.1	0.01	0.0
0.2	0.04	0.4
0.3	0.09	0.7
0.4	0.16	1.3
0.5	0.25	1.9
0.6	0.36	2.9
0.7	0.49	3.9
0.8	0.64	5.1
0.9	0.81	6.6
1.0	1.00	8.1

In this case, if we plot a graph of power loss against current, we expect a parabola.



From this graph it is not possible to determine (easily) the constant of proportionality. However, this problem can be overcome if a graph of the current squared is plotted against the power loss.



Now the constant of proportionality can be found from the graph

$$R = \frac{\Delta P}{\Delta I^2} = \frac{8.0}{1.0} = 8.0$$

Therefore, the relationship is

$$P = 8.0I^2$$

These ideas can be extended to other non-linear relationships, enabling those non-linear relationships to be reduced to a linear form, from which the constant of proportionality and the intercept can be determined.

For example, for relationships of the form

$$\frac{1}{y} + \frac{k}{x} = \frac{1}{a}$$

rewrite them as

$$\frac{1}{y} = -k\frac{1}{x} + \frac{1}{a}$$

and by comparing it with the equation of the straight line

$$Y = mX + c$$

plot

$$Y = \frac{1}{y} \text{ against } X = \frac{1}{x}$$

Then the gradient of this graph will give the constant of proportionality

$$-k = m = \frac{\Delta Y}{\Delta X}$$

and the intercept will be

$$c = \frac{1}{a}$$

Log/linear relationships

Relationships of the form

$y = ax^n$

can also be reduced to linear form, by first taking logarithms. This transforms the non-linear equation to a linear form

$\log y = n \log x + \log a$

By comparing with

$Y = mX + c$

plot

$Y = \log y$ against $X = \log x$

Then the gradient of this graph will give the value of the index in the equation

$$n = \frac{\Delta Y}{\Delta X} = \frac{\Delta \log y}{\Delta \log x}$$

and the intercept will be

$c = \log a$

Example

Some molecules are made out of two atoms. The moment of inertia and the distance between the nuclei of the atoms is given for four such molecules in the table bellow

Moment of inertia I $\left(10^{-40}\ \text{g}\cdot\text{cm}^2\right)$	1.34	2.66	3.31	4.31
Distance between nuclei r $\left(10^{-8}\ \text{cm}\right)$	0.92	1.28	1.42	1.62

Find a law in the form $I = k \cdot r^n$.

Solution

$I = k\,r^n$

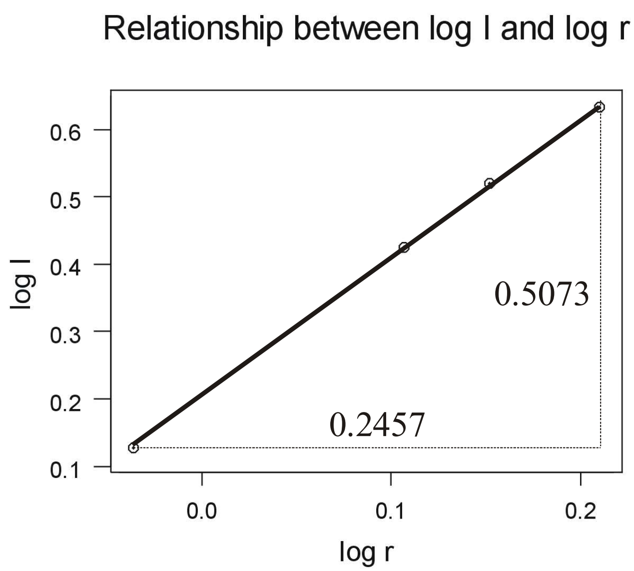
Taking logarithms to the base 10 of each side, we have

$$\log_{10}(I) = n\log_{10}(r) + \log_{10}(k)$$

Transforming the values respectively

I	r	$\log I$	$\log r$
1.34	0.92	0.127	–0.036
2.66	1.28	0.425	0.107
3.31	1.42	0.520	0.152
4.31	1.62	0.634	0.210

Plotting the graph of $\log_{10}(I)$ against $\log_{10}(r)$ gives



From the graph we obtain the gradient of the line $Y = nX + \log_{10}(k)$

$$\begin{aligned} n &= \frac{\log_{10}(4.31) - \log_{10}(1.34)}{\log_{10}(1.62) - \log_{10}(0.92)} \\ &= \frac{0.6344 - 0.1271}{0.2095 + 0.0362} \\ &= \frac{0.5073}{0.2457} = 2.0647 = 2 \text{ (nearest integer value)} \end{aligned}$$

Also

$$k = \frac{1.34}{(0.92)^2} = 1.5831 \approx 1.6$$

Therefore, on substituting into $I = k\,r^n$, we obtain

$$I = 1.6 \times r^2$$

Further log/linear relationships

Relationships of the form

$$y = ab^x$$

are likewise transformed to linear form by taking logarithms. This gives

$$\log y = x \log b + \log a$$

By comparing with

$$Y = mX + c$$

plot

$Y = \log y$ against $X = x$

Then the gradient of this graph will give the value of b in the equation

$$b = \frac{\Delta Y}{\Delta X} = \frac{\Delta \log y}{\Delta x}$$

and the intercept will be

$$c = \log a$$