Logarithms and Implicit Differentiation

Prerequisites

You should be familiar with the rules for the differentiation of exponential and logarithmic functions and with the technique of implicit differentiation. Naturally, you should also be able to use the Leibniz and chain rules for the differentiation of products and composites of functions, and be familiar with stationary points and how to test for them, both by means of the second derivative and by looking at how the sign of the derivative changes around the stationary point.

Example (1)

By differentiating $e^{2x}\ln y = 2x$ implicitly show that $\frac{dy}{dx} = \frac{2e^{\left(\frac{2x}{e^{2x}}\right)}}{e^{2x}}(1-2x)$.

Solution

$e^{2x}\ln y = 2x$

$2e^{2x}\ln y + e^{2x}\frac{1}{y}\times\frac{dy}{dx} = 2$

$\frac{e^{2x}}{y}\times\frac{dy}{dx} = 2\left(1-e^{2x}\ln y\right)$

$\frac{dy}{dx} = \frac{2y}{e^{2x}}\left(1-e^{2x}\ln y\right)$

$\frac{dy}{dx} = \frac{2y}{e^{2x}}\left(1-e^{2x}\times 2xe^{-2x}\right)$ $\qquad \left[\ln y = 2xe^{-2x}\right]$

$\frac{dy}{dx} = \frac{2e^{\left(\frac{2x}{e^{2x}}\right)}}{e^{2x}}(1-2x)$ $\qquad \left[y = e^{2xe^{-2x}} = e^{\left(\frac{2x}{e^{2x}}\right)}\right]$

Further derivatives of exponential and logarithmic functions

We use implicit differentiation to find the derivative of $y = 2^x$ as follows

$y = 2^x$

$\ln y = \ln\left(2^x\right) = x\ln 2$ $\qquad$ [Taking the log of both sides]

$\frac{1}{y}\times\frac{dy}{dx} = \ln 2$ $\qquad$ [Differentiating implicitly]

$\frac{dy}{dx} = y\ln 2 = \ln 2\times 2^x$ $\qquad$ [Rearranging and substituting back]

Example (2)

Find the derivative of $y = x^{\ln x}$ where $x > 0$. What are the coordinates of its stationary point? What kind of stationary point is it? Sketch the curve of $y = x^{\ln x}$.

Solution

$y = x^{\ln x}$

$\ln y = \ln\left(x^{\ln x}\right)$

$\ln y = \ln x \cdot \ln x = \left(\ln x\right)^2$

$\frac{1}{y} \cdot \frac{dy}{dx} = 2\ln x \cdot \frac{1}{x} = \frac{2}{x}\ln x$

$\frac{dy}{dx} = x^{\ln x} \cdot \frac{2}{x}\ln x$

For turning points $\frac{dy}{dx} = 0$, so $x^{\ln x} \cdot \frac{2}{x}\ln x = 0$

Since $x^{\ln x}$ and $\frac{2}{x}$ are always positive on the domain $x > 0$, then

$\ln x = 0 \Rightarrow x = 1 \Rightarrow y = 1^{\ln 1} = 1^0 = 1$

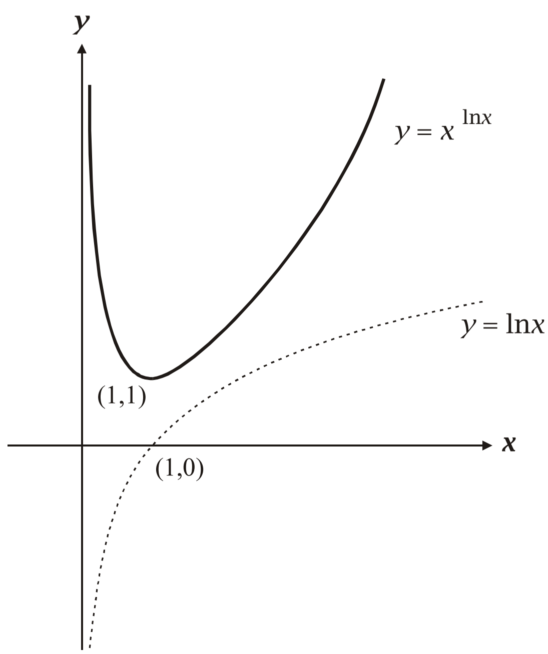
To find the character of the stationary point let us see how $\frac{dy}{dx}$ changes around $x = 1$

$x < 1 \quad \ln x < 0 \Rightarrow \frac{dy}{dx} < 0 \qquad x = 1 \quad \frac{dy}{dx} = 0 \qquad x > 1 \quad \ln x > 0 \Rightarrow \frac{dy}{dx} > 0$

So this is a minimum.

The graph of $y = x^{\ln x}$ is asymptotic to the y–axis; that is $x \to 0 \quad y \to \infty$

Also $x \to \infty \quad y \to \infty$



Example (3)

Find the derivative of $y = (\sin x)^x$ where $0 < x < \pi/2$.

Solution

$$y = (\sin x)^x$$
$$\ln y = x \ln(\sin x)$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(\sin x) + x \cdot \frac{1}{\sin x} \cdot \cos x$$
$$\frac{dy}{dx} = y\left(\ln(\sin x) + x \cot x\right) = (\sin x)^x \left(\ln(\sin x) + x \cot x\right)$$

Example (4)

Find the derivative of $y = \dfrac{(x^2+3)^{\frac{1}{2}}}{(5\sin x - 1)^{\frac{2}{3}}}$

Solution

$$y = \frac{(x^2+3)^{\frac{1}{2}}}{(5\sin x - 1)^{\frac{2}{3}}}$$
$$\ln y = \ln\left\{\frac{(x^2+3)^{\frac{1}{2}}}{(5\sin x - 1)^{\frac{2}{3}}}\right\} = \frac{1}{2}\ln(x^2+3) - \frac{2}{3}\ln(5\sin x - 1)$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2(x^2+3)} \cdot 2x - \frac{2}{3} \cdot \frac{1}{(5\sin x - 1)} \cdot (5\cos x)$$
$$\frac{dy}{dx} = y\left\{\frac{x}{x^2+3} - \frac{10\cos x}{3(5\sin x - 1)}\right\} = \left\{\frac{(x^2+3)^{\frac{1}{2}}}{(5\sin x - 1)^{\frac{2}{3}}}\right\}\left\{\frac{x}{x^2+3} - \frac{10\cos x}{3(5\sin x - 1)}\right\}$$