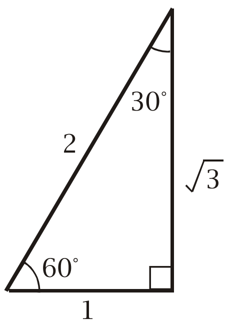
Maclaurin Series

The need for approximating functions

The functions e^x and $\sin x$ are examples of *transcendental functions.* You should be familiar with functions like e^x and $\sin x$, and also with finding values such as $e^{3.7}, \sin\left(1/7\right)$ by using a calculator or from tables. But how are these values arrived at in the first place? In certain cases we know the *exact* value, such as $\cos\left(\pi/3\right) = 1/2$. This is because we can construct a "special triangle" which shows this to be exact.



But this does not help with $\sin\left(1/7\right)$. The function $f(x) = x^3 + 2x^2 - x + 5$ is an example of a *polynomial function.* It would be helpful if we could write a function like $\sin x$ as a polynomial, but this *cannot be done!* That is precisely why functions like $\sin x$ are said to be *transcendental functions.* However, there is a way around this problem. We *can* write $\sin x$ as an *infinite* series of polynomial functions and we can use a *finite* part of such a series to find the value of $\sin x$ for given x to any required level of accuracy. To make this clearer, the series expansion for $\sin x$ is in fact

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

The dots mean that the series go on forever, according to a pattern. So, the next term in this series would be $+\frac{x^9}{9!}$. If we take a finite part of this we obtain an approximation for $\sin x$. Thus, for example, an approximation for $\sin\left(1/7\right)$ is

$$\sin\left(1/7\right) = \frac{1}{7} - \frac{\left(\frac{1}{7}\right)^3}{3!} + \frac{\left(\frac{1}{7}\right)^5}{5!} = 0.14237173$$

Comparing this with the calculator value of 0.142371729... indicates that (a) whilst it is a good approximation it is still only an approximation and therefore (b) more terms should be added to get closer and closer to the true value. From this example, we can see here that simply having the series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ does not solve all the problems. To be useful (a) we need to know that the series *converges* on the exact real value of $\sin x$ and (b) we need to know at any finite stage how close we are to that real value – in other words, what the difference is between the real value and the approximation. However, these two topics of (a) convergence and (b) error are advanced topics and are left for a later chapter. The purpose of this discussion is simply to show you *why* we need to approximate functions by series of polynomial functions. We are also going to adopt a "cookery book" approach here to the subject – meaning, that we will show you the general formula and how to apply it, rather than prove the formula, since this is also a more advanced topic.

Regarding $\sin x$, note that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ really provides us with a series of approximating polynomial functions.

$p_1(x) = x$

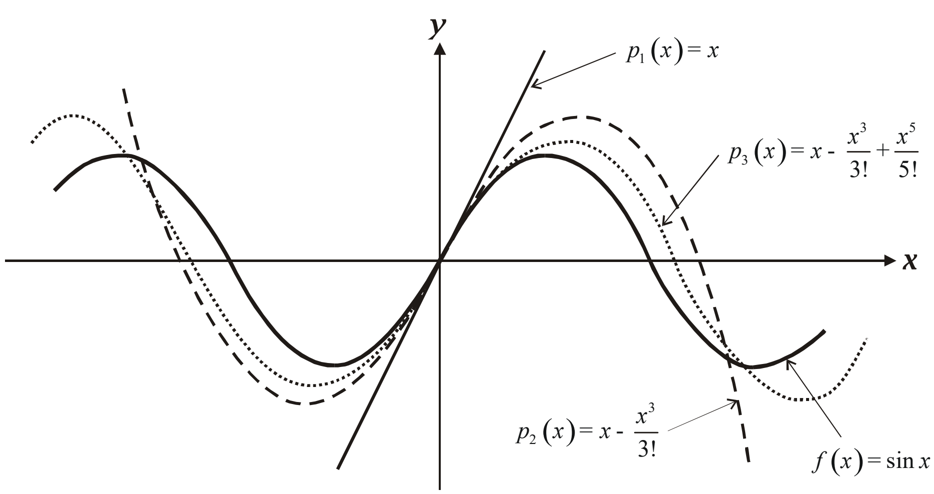
$p_2(x) = x - \frac{x^3}{3!}$

$p_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$p_4(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

Let us illustrate what is meant by an approximation by looking at the graphs of some of these.



Each successive approximation in this case is introducing an extra pair of minima and maxima. The further the value of x is taken away from 0 the worse the approximation. The graph also shows that the approximation is centred around 0. The expression $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ is said to be the standard series approximation for $\sin x$ about 0. It is called the Maclaurin series for $\sin x$, after the Scottish mathematician Colin Maclaurin (1698 – 1796). His theorem is called Maclaurin's Theorem, and it has been vitally important to the development of mathematics because it enables transcendental functions such as e^x and $\sin x$ etc., to be approximated by polynomial functions. Transcendental functions are not directly calculable, but polynomial functions are.

Maclaurin's Theorem

A function $f(x)$ can be approximated by an infinite series of polynomial terms in *x*. The approximation fits around the graph of $f(x)$ about a fixed point. In Maclaurin's Theorem the fixed point is 0. The polynomial approximation about 0 is

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots + \frac{f^{(n)}(0)x^n}{n!} + \ldots. \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} \end{aligned}$$

In this equation the symbols $f', f'', f''', \ldots f^{(n)}, \ldots$ mean the first, second, third and *n*th derivative of $f(x)$ respectively. The symbol $f^{(n)}(0)$ means the *n*th derivative of $f(x)$ evaluated at 0. We can see from the use of these symbols that for the theorem to apply the function must be infinitely differentiable. The series must also converge on the function $f(x)$. In other words, each successive term in the series must get smaller and smaller. This is written

$$\frac{f^{(n)}(0)}{n!} \to 0 \ \text{ as } \ n \to 0$$

At this level we will simply assume that all the functions for which we are asked to derive Maclaurin series do have this convergence property.

Example (1)

Find the standard Maclaurin series for $\sin x$ up to the term in x^7.

Solution

The formula is

$$f(x)=f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\ldots+\frac{f^{(n)}(0)x^n}{n!}+\ldots.$$

This requires us to find the derivatives of $\sin x$ up to the seventh derivative and to evaluate each at 0.

$f(x)=\sin x$	$f(0)=\sin 0=0$
$f'(x)=\cos x$	$f'(0)=\cos 0=1$
$f''(x)=-\sin x$	$f''(0)=-\sin 0=0$
$f'''(x)=-\cos x$	$f'''(0)=-\cos 0=-1$
$f^{(4)}(x)=\sin x$	$f^{(4)}(0)=\sin 0=0$
$f^{(5)}(x)=\cos x$	$f^{(5)}(0)=\cos 0=1$
$f^{(6)}(x)=-\sin x$	$f^{(6)}(0)=-\sin 0=0$
$f^{(7)}(x)=-\cos x$	$f^{(7)}(0)=-\cos 0=-1$

On substituting into

$$f(x)=f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\ldots+\frac{f^{(n)}(0)x^n}{n!}+\ldots.$$

we get

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

Note that the series shows a definite pattern and that by the second derivative we have

$$f''(x)=-\sin x=-f(x)$$

In other words the second derivative can be written in terms of the first derivative. This can make finding series expansions easier. For example we could have written

$$f(x)=\sin x$$
$$f'(x)=\cos x$$
$$f''(x)=-\sin x=-f(x)$$
$$f'''(x)=-f'(x)$$
$$f^{(4)}(x)=f''(x)=f(x)$$
$$f^{(5)}(x)=-f(x)$$

This makes evaluating the series much simpler.

Example (2)

Find the Maclaurin series for $f(x)=\cos 3x$ up to the term in x^4.

Solution

$f(x) = \cos 3x$ $\qquad f(0) = 1$

$f'(x) = -3\sin 3x$ $\qquad f'(0) = 0$

$f''(x) = -9\cos 3x$ $\qquad f''(0) = -9$

$f^{(3)}(x) = 27\sin 3x$ $\qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = 81\cos 3x$ $\qquad f^{(4)}(0) = 81$

Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots$$

Therefore, on substituting $\cos 3x$ for $f(x)$ and replacing each term by its corresponding value

$$\cos 3x = 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} + \ldots$$
$$= 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \ldots$$

Hence

$$\cos 3x \approx 1 - \frac{9x^2}{2} + \frac{27x^4}{8}$$

Example (3)

Find the Maclaurin series for $f(x) = \ln(1+4x)$ up to the term in x^4. Use your series to find an approximation to $\ln(1.04)$ to 6 d.p.

$f(x) = \ln(1+4x)$ $\qquad f(0) = \ln(1) = 0$

$f'(x) = \frac{1}{1+4x} \times 4 = 4(1+4x)^{-1}$ $\qquad f'(0) = \frac{4}{1} = 4$

$f''(x) = -16(1+4x)^{-2}$ $\qquad f''(0) = -16$

$f^{(3)}(x) = 128(1+4x)^{-3}$ $\qquad f^{(3)}(0) = 128$

$f^{(4)}(x) = -1536(1+4x)^{-4}$ $\qquad f^{(4)}(0) = -1536$

Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots$$

Therefore, on substituting $\ln(1+4x)$ for $f(x)$ and replacing each term by its corresponding value

$$\ln(1+4x) = 0 + \frac{4x}{1!} - \frac{16x^2}{2!} + \frac{128x^3}{3!} - \frac{1536x^4}{4!}$$
$$= 4x - 8x^2 + \frac{64x^3}{3} - 64x^4$$
$$\ln(1.04) = \ln(1 + 4 \times 0.01)$$
$$\approx 4(0.01) - 8(0.01)^2 + \frac{64(0.01)^3}{3} - 64(0.01)^4$$
$$= 0.039221 \text{ (6 d.p.)}$$

Standard Maclaurin series

There are a number of standard Maclaurin series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

It is necessary to give an expansion for $\ln(1+x)$ rather than $\ln x$ because there is no Maclaurin series for $\ln x$. This is because, when $f(x) = \ln(x)$ then $f'(x) = 1/x$. Hence, when evaluating at $x = 0$, we are required to divide by 0, and that is not permissible.

From the Maclaurin series we can also derive the small angle approximations for $\sin x$, $\cos x$ and $\tan x$, which are

$$\sin x \approx x$$
$$\cos x \approx 1 - \frac{x^2}{2}$$
$$\tan x \approx x.$$

These arise from taking only the first non-zero term in x. These approximations only apply when x is small.[1]

[1] However, logically the result $\sin x \approx x$ must be shown independently of the proof of Maclaurin's series, since $\frac{x}{\sin x} \to 1$ as $x \to 0$ is used to *prove* that $\frac{d}{dx}\sin x = \cos x$, which is then required when finding the Maclaurin series for $\sin x$.

Taylor series

We indicated that the Maclaurin series are polynomial approximations to a function $f(x)$ about 0. That is, the value of $f(0)$ is fixed and the polynomial function is centred on this value. The disadvantage of this is that as we move further and further away from $x = 0$ the approximation to $f(x)$ gets worse and worse. So sometimes we need to centre the approximation on a different value, when $x = a$. For historic reasons an approximation centred on a value $x = a$ where a is some other value than zero, is called a Taylor series after the English mathematician Brook Taylor (1685 – 1731).

The Maclaurin series, which is the polynomial approximation about 0 to a function $f(x)$, is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots + \frac{f^{(n)}(0)x^n}{n!} + \ldots.$$

The Taylor series, which is the polynomial approximation about a to a function $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)x^n}{n!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \ldots.$$

Example (4)

Find the Taylor series for $f(x) = \dfrac{1}{(1+x)}$ at $x = 1$ up to the term in x^4.

$$f(x) = (1+x)^{-1} = \frac{1}{(1+x)} \qquad f(1) = \frac{1}{2}$$

$$f'(x) = -(1+x)^{-2} = -\frac{1}{(1+x)^2} \qquad f'(1) = -\frac{1}{4}$$

$$f''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3} \qquad f''(1) = \frac{1}{4}$$

$$f^{(3)}(x) = -6(1+x)^{-4} = -\frac{6}{(1+x)^4} \qquad f^{(3)}(1) = -\frac{3}{8}$$

$$f^{(4)}(x) = 24(1+x)^{-5} = \frac{24}{(1+x)^5} \qquad f^{(4)}(1) = \frac{24}{32} = \frac{3}{4}$$

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \frac{f^{(4)}(a)(x-a)^4}{4!} \\ &= f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^{(4)}(1)(x-1)^4}{4!} \\ &= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{4} \times \frac{1}{2!}(x-1)^2 - \frac{3}{8} \times \frac{1}{6}(x-1)^3 + \frac{3}{4} \times \frac{1}{24}(x-4)^4 \\ &= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-4)^4 \end{aligned}$$