Mappings of complex numbers: Transformations from the z to the w plane

Prerequisites

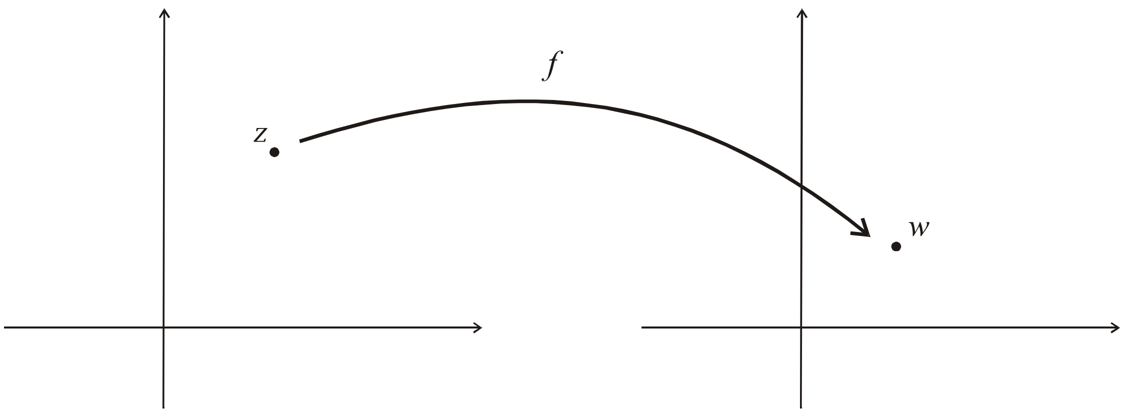
You should be familiar with the algebra of complex numbers up to elementary complex loci.

Complex functions

Complex functions that map from one complex number to another cannot be represented in a two-dimensional graph, because both the domain and the co-domain are themselves two dimensional (being represented by the Argand plane), so the mapping itself is four-dimensional. To provide a visual idea of the effect of a complex function, we use a *mapping diagram*. The image of the point z in the domain is usually denoted by w. Then the complex function, f, is given by

$$f\begin{cases}\mathbb{C} \to \mathbb{C} \\ z \mapsto w\end{cases}$$

We also write $w = f(z)$. The mapping diagram is



These mappings are also called transformations of the z to the w plane.

Geometric interpretation of complex functions

In fact, by this stage, you are already familiar with several complex functions, and we now show how these can be given a geometric interpretation using the mapping diagram.

Translation by a scalar

The mapping

$w = f(z) = z + c$

where c is a complex number, represents a translation through c.

Example (1)

Plot the effect of the mapping $f(z) = z + (3+i)$

$w = f(z) = z + a$

on the line with locus $\arg z = \frac{\pi}{4}$.

Solution

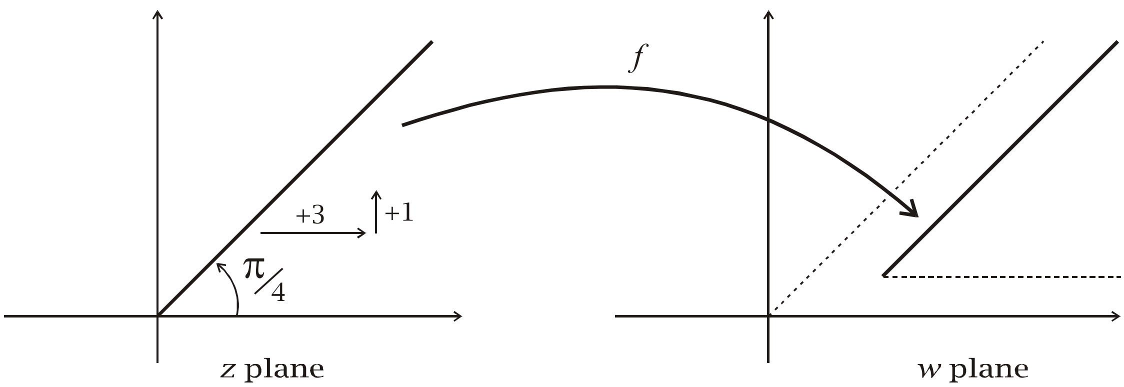
The line with locus $\arg z = \frac{\pi}{4}$ has Cartesian equation $y = x$ but is restricted to the domain $x \geq 0$. The complex number $(3+i)$ translates this line 3 along and 1 up. It could be represented by a vector thus

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+3 \\ y+1 \end{pmatrix}$

or by

$w = (x+iy) + (3+i) = (3+x) + i(y+1)$

The mapping diagram is



Enlargement by scale factor *k*

The mapping

$w = c\,z$

where k is a complex number, represents an enlargement by scale factor | c | and a rotation anti-clockwise through arg c.

To show this we place c and z into polar form; then

$\text{Let}\, c = \left[\,|c|, \arg c\right]$ and $z = \left[\,|z|, \arg z\right]$

then by the rules for multiplication of complex numbers in polar form (multiply the moduli, add the arguments)

$c\,z = \left[\,|c||z|, \arg c + \arg z\right]$

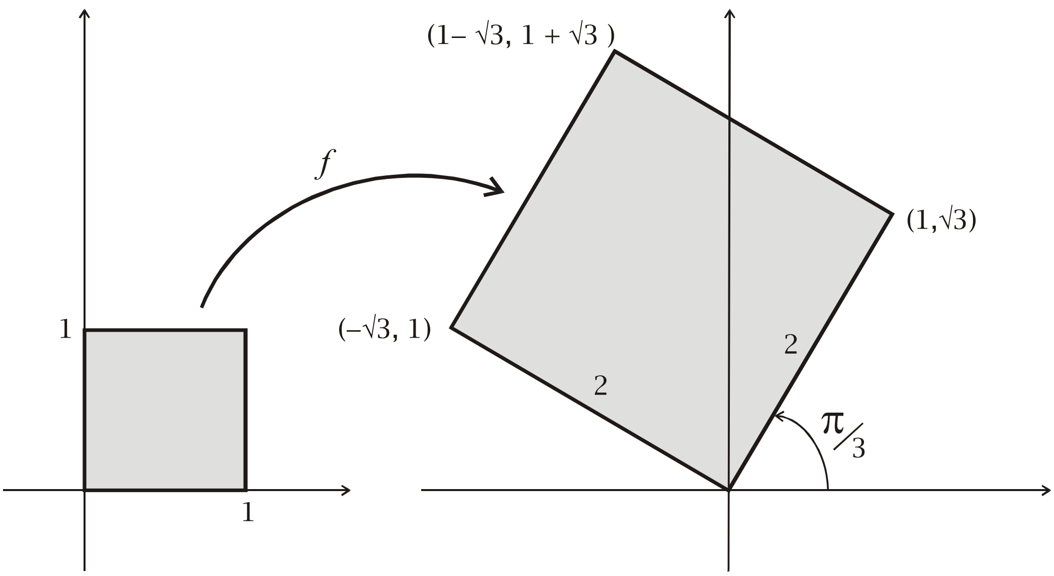
Example (2)

Plot the effect of $w = \left(1+\sqrt{3}i\right)z$ on the region of the z plane with locus

$0 \le \operatorname{Re} z \le 1$
$0 \le \operatorname{Im} z \le 1$

Solution

Here $c = \left(1+\sqrt{3}i\right)$ and $|c| = 2, \arg c = \dfrac{\pi}{3}$ so the mapping diagram is



Reflection in the real axis

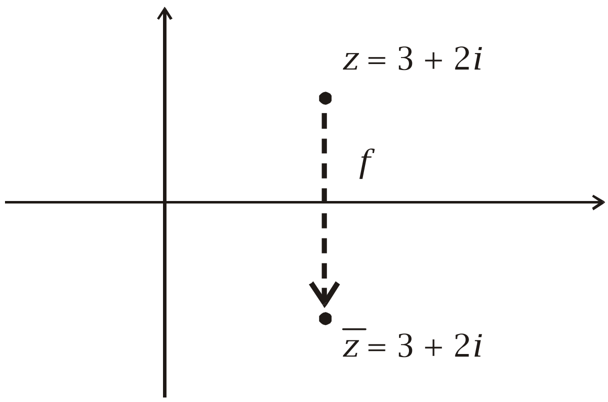
The operation of taking the complex conjugate, equivalent to the mapping

$w = \bar{z}$

represents a reflection in the real axis.

Example (3)
Plot on the same Argand diagram the complex numbers z and $\bar{z}$ where
$z = (3 + 2i)$

Solution



Composition of mappings

When one complex function is followed by another, this is the composition of mappings. Just as with real functions we can visualise the effect of the composition of functions in terms of one transformation followed by another. We illustrate this idea by means of a worked example.

Example (4)
Determine the effect of the mapping
$z \mapsto -i\left(z - (2 + i)\right)$

on the locus given by $\arg(z - 1) = \dfrac{\pi}{3}$.

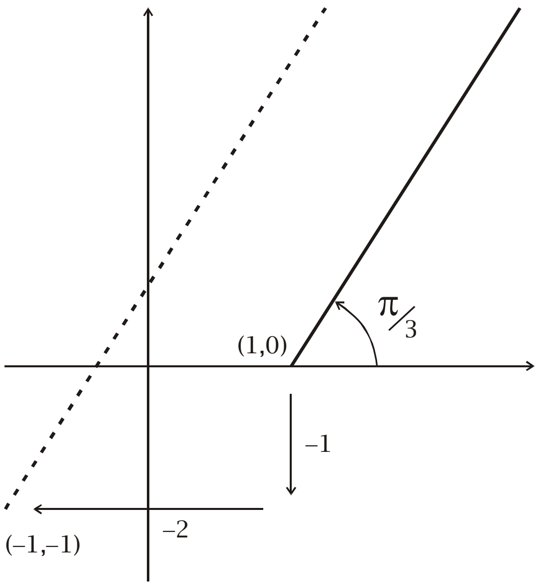
Solution

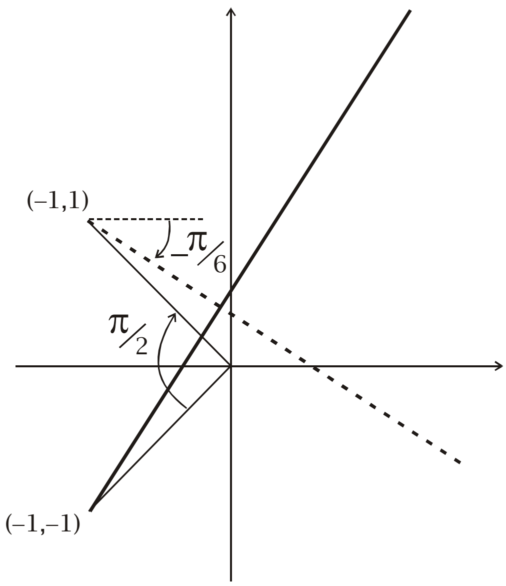
The locus $\arg(z - 1) = \dfrac{\pi}{3}$ is the straight-line segment beginning at 2 with angle $\dfrac{\pi}{3}$. This line is first translated by
$z \mapsto z - (2 + i)$

that is, –1 in the x direction, and –1 in the y direction. The resultant line is then rotated by $\frac{\pi}{2}$ in the *clockwise* direction (clockwise, since it is multiplied by $-i$. This is the transformation

$z \mapsto -iz$

Graphically, the effect is as follows





The locus of the image is

$\arg\left(z-\left(-1+i\right)\right)=-\frac{\pi}{6}$

Finding transformations of loci by means of substitution

We will now solve the last problem algebraically by means of the substitution

$z=x+iy$

The image of z under the transformation $z \mapsto -i\left(z-\left(2+i\right)\right)$ will be

$w=u+iv$

Second solution to example (4)

The line given by $\arg\left(z-1\right)=\frac{\pi}{3}$ has Cartesian equation

$y=\sqrt{3}x-\sqrt{3} \qquad x \geq 1$

To solve this problem we first substitute $z = x + iy$ into $-i(z-(2+i))$. This gives

$$\begin{aligned} w &= -i(z-(2+i)) \\ &= -i((x+iy)-(2+i)) \\ &= -i((x-2)+i(y-1)) \\ &= (y-1)+(2-x)i \end{aligned}$$

That is

$u = y - 1 \qquad v = 2 - x$

At this stage we are seeking a relationship between v and u of the form

$v = f(u)$

so we need to eliminate x and y from this equation by means of algebra. We have the equation

$y = \sqrt{3}x - \sqrt{3} \qquad x \geq 1$

We want to substitute from this into $v = 2 - x$ so we need to rearrange it to get x the subject.

$x = \frac{1}{\sqrt{3}}y + 1 \qquad y \geq 0$

Then

$$\begin{aligned} v &= 2 - x \\ &= 2 - \left(\frac{1}{\sqrt{3}}y + 1\right) = 1 - \frac{1}{\sqrt{3}}y \end{aligned}$$

Now we have to eliminate y from this by substituting from $u = y - 1$; that is, $y = u + 1$.

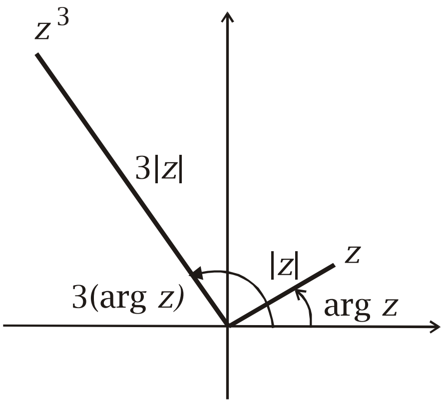
Hence

$$\begin{aligned} v &= 1 - \frac{1}{\sqrt{3}}y && y \geq 0 \\ &= 1 - \frac{1}{\sqrt{3}}(u+1) && u \geq 1 \\ &= -\frac{1}{\sqrt{3}}u + \left(1 - \frac{1}{\sqrt{3}}\right) && u \geq 1 \end{aligned}$$

This agrees with our previous solution.

The mapping $z \mapsto z^n$

Consider $z = [|z|, \arg z] = [r, \theta]$ and let us plot z^3. Then graphically we plot z^3 by observing that the argument of z^3 is 3 times the argument of z and the modulus of z^3 is the cube of the modulus of z



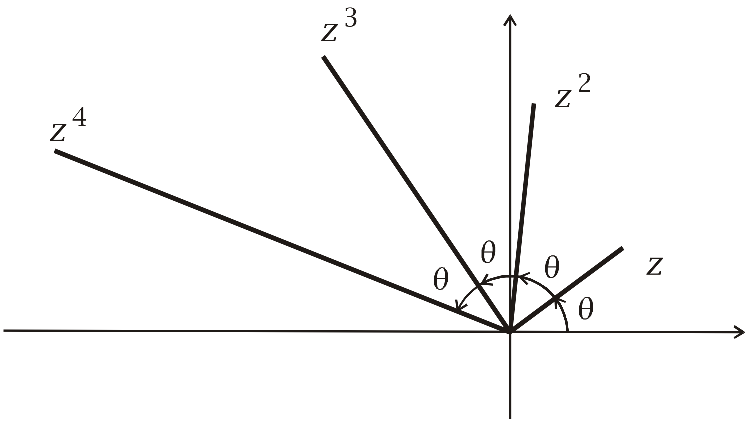
That is

$z^3 = \left[|z|^3, 3\arg z \right] = \left[r^3, 3\theta \right]$

This observation automatically suggests that for all n

$z^n = \left[|z|^n, n\arg z \right] = \left[r^n, n\theta \right]$

This conjecture is in fact correct and is known as *De Moivre's Theorem.* It is the subject of another chapter. If $r = |z| > 1$ then the values of z^2, z^3, z^4 …. "spiral outwards":



If $r < 1$ then the values of z^2, z^3, z^4 …. "spiral inwards"

If $r = 1$ then the values of $z^2, z^3, z^4, \ldots$ all lie on the unit circle.

Example (5)

A complex function is given by

$f : w = z^2$

Prove that the image under f of the line with parametric equation

$t \mapsto t + ci$

where c is a constant is a parabola. Sketch this locus when $c > 1$

Solution

Substitute $z = t + ic$

into f to get

$$u + iv = (t + ic)^2$$
$$= t^2 - c^2 + 2cti$$

This gives

$$u = t^2 - c^2 \qquad (1)$$
$$v = 2ct \qquad (2)$$

From (2)

$$v^2 = (2c)^2 t^2$$
$$t^2 = \frac{v^2}{(2c)^2}$$

and substituting this in (1)

$$u = \frac{v^2}{(2c)^2} - c^2$$

which is the equation of a parabola with axis of symmetry the x-axis and vertex at $-c^2$.

We can also factorise this as

$$u = \frac{1}{(2c)^2}(v - 2c^2)(v + 2c^2)$$

so the parabola cuts the v axis at $\pm 2c^2$. For $c > 1$ a sketch of the locus is