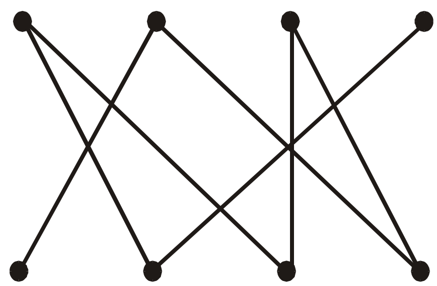
Matching Problems

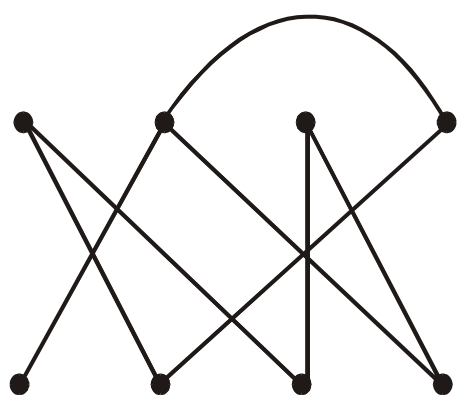
Bipartite graphs

A bipartite graph is a graph where the nodes (vertices) are divided into two sets. Edges (arcs) may connect nodes from one set to the other, but edges are not allowed to connect nodes within the sets.

This graph



is a bipartite graph, but this one

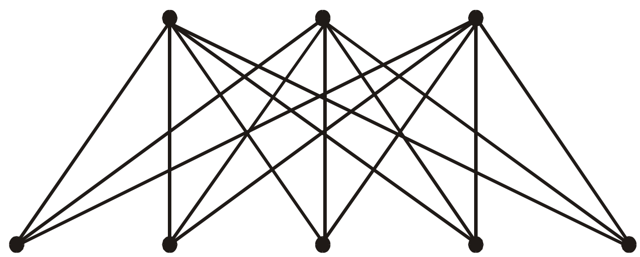


is not bipartite.

In general, a bipartite graph can be made from two sets of unequal size.

A complete bipartite graph is one where each node (vertex) of one set is connected to every node (vertex) of the other. Suppose we label the two subsets of the bipartite graph *M* and *N*, and suppose the size of *M* is *m* and the size of *N* is *n*. Then the complete bipartite graph is denoted $K_{m,n}$.

For example, the bipartite graph $K_{3,5}$ is



Matching problems

Bipartite graphs are of use as a tool when solving matching problems. Matching problems are problems where one is required to pair off elements from one set with elements from the other. We will consider situations where the two sets are equal in size, and a valid solution to the problem is any pairing of elements from one set with elements from the other such that every element from the first set is paired with one and only one other set. Sometimes more than one solution to the problem may be possible. On other occasions it may not be possible to solve the problem.

The process is best illustrated by example.

<u>Example</u>

A charity has decided to give five presents to orphans in Romania. They want to ensure that each orphan gets a present that they really would like, and they have devised a ploy, using an Englishman impersonating Father Christmas, to find out the children's secret wishes.

The five presents are (1) an electric train, (2) a gameboy, (3) a cricket set, (4) mechano and (5) a chemistry set.

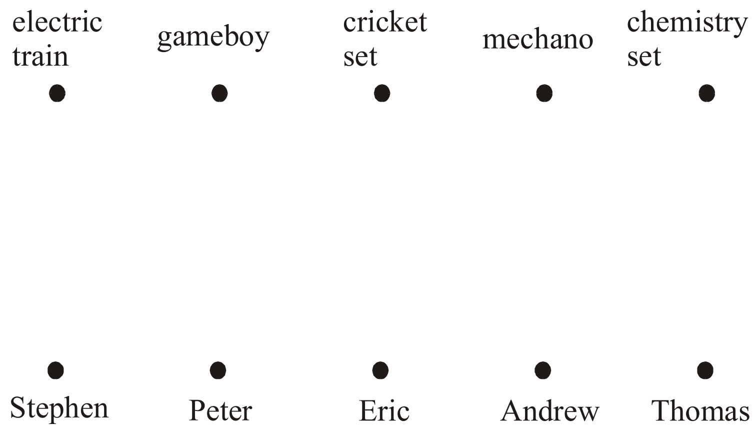
Stephen has secretly wished for an electric train or mechano; Peter has secretly wished for an electric train, a cricket set or a chemistry set; Eric has made a secret wish for an electric train, a cricket set or a chemistry set; Andrew has secretly wished for a gameboy or mechano; Thomas has secretly wished for a gameboy.

(*i*) Draw a graph representing this matching problem.

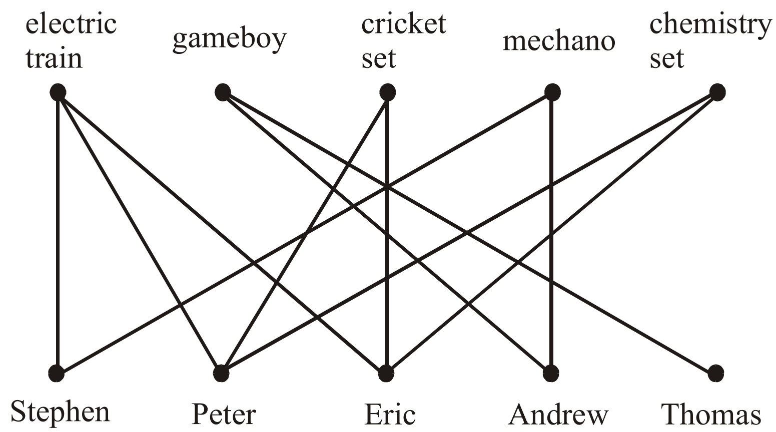
(*ii*) By constructing an alternating path find a suitable matching so that each child does obtain his secret wish.

Solution

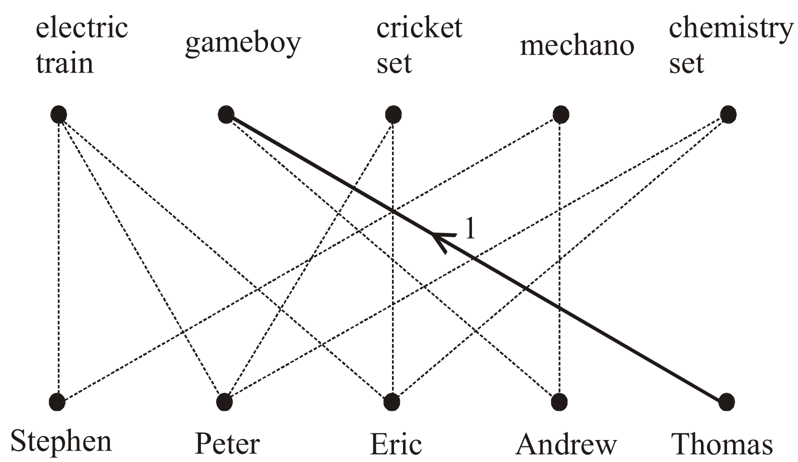
We begin by representing the elements of the two sets as nodes, and place a clear space between the two sets of nodes.



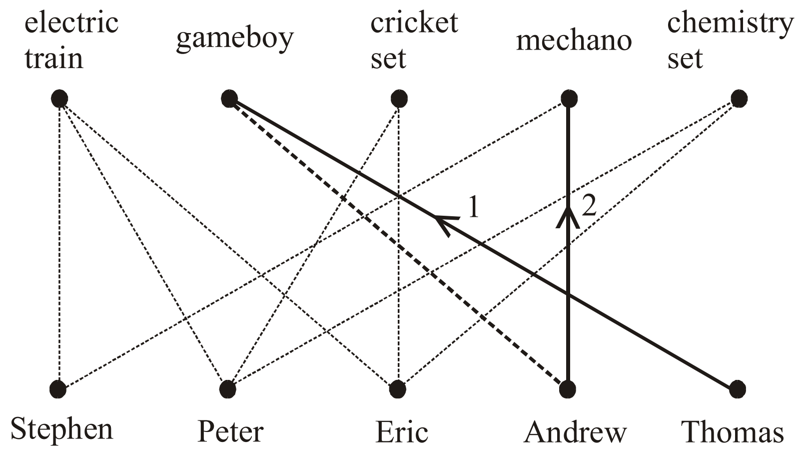
The next stage is to draw an edge joining any *possible* pairing of the elements from the first set with the elements of the second.



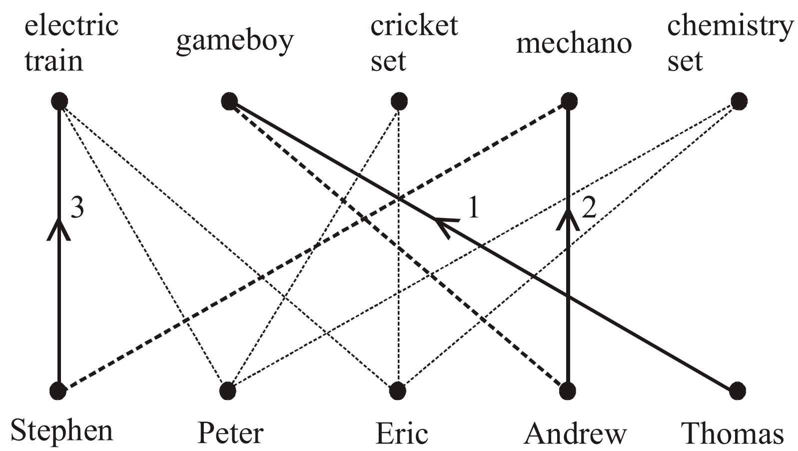
Now we use the bipartite graph as an aid to the solution of the problem. Looking at the graph indicates that we must assign the gameboy to Thomas, so we begin there.



To solve the problem we alternate between the two sets. Having assigned the gameboy to Thomas there is only one way to return to the set of the orphans, and that is to Andrew. Andrew must then be given the mechano.

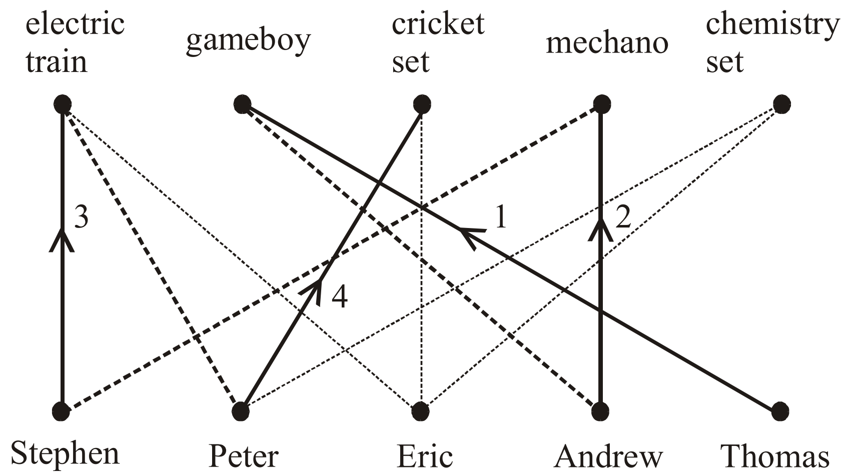


Once again, we can only return to the set of children by moving next to Stephen, and Stephen must then have the electric train.



At this point we have a *choice* of which way to return to the set of children, we can move either to Peter or to Eric. One of these choices may not lead to a solution, so we would have to make a note of this point, and the choice made, so that if a solution is not found we could return to it and try the other. If no choices at such a branch point lead to a possible solution, then we conclude that the matching problem is not soluble.

In this case, both choices would lead to a possible solution. We choose one at random; for example, we choose to return to Peter and assign to him the cricket set.



The final assignment gives the chemistry set to Eric.