Mathematical Induction

Prerequisites

You should be familiar with the summation notation and the method of summing finite series using standard results. We remind you, however, that a series is an expression of the form

$$\sum_{i=1}^{n} u_i$$

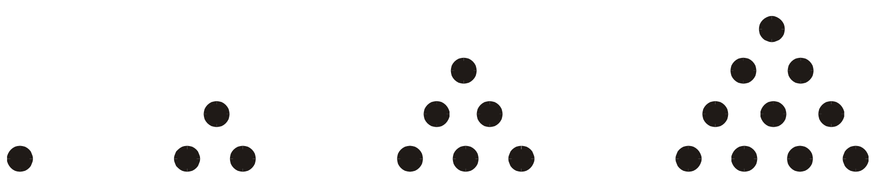
For example, $u_i = i$

$$S_n = \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n$$

Mathematical induction is a unique process of mathematical thinking and is a form of proof. In mathematical induction we prove the validity of a general proposition from just two statements. In other words, it is a form of reasoning that enables the mathematician to draw conclusions about infinite series from finite reasoning. So it has to be special! A general proposition is a statement that applies to all numbers. The general proposition is usually guessed at, in the first instance, by trial and error, leading to a conjecture. So firstly, we illustrate how a conjecture might arise.

Number patterns and conjectures

Consider the following diagrams representing a sequence of numbers.



1 3 6 10

It is clear that the sequence of numbers could be continued indefinitely. The sequence is a mapping from the nth number to the nth value of the number.

$1 \to 1$
$2 \to 3$
$3 \to 6$
$4 \to 10$

The question is: what does n map to and how might this nth value be shown to be the nth value for certain? The diagram above indicates that

$1 \to 1$
$2 \to 1+2=3$
$3 \to 1+2+3=6$
$4 \to 1+2+3+4=10$

Hence the nth term in the series is

$$S_n = \sum_{i=1}^{n} i = 1+2+3+\ldots+n$$

This is "the sum of the n terms from 1 to n." The triangular representation indicates how we arrive at a conjecture of what this sum might be. For example, for $n = 4$, suppose we make a copy of the triangle and join the two triangles together:

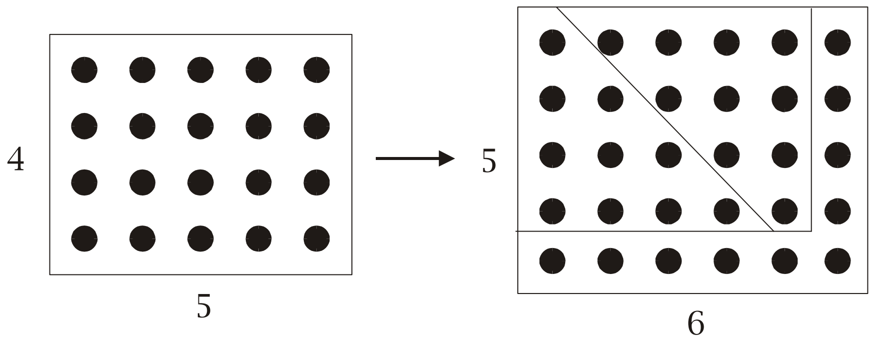
The result is a 4×5 rectangle. This rectangle is twice the size of the 4th triangle. This suggests the conjecture that the nth triangle is half the size of the $n(n+1)$ th rectangle

CONJECTURE

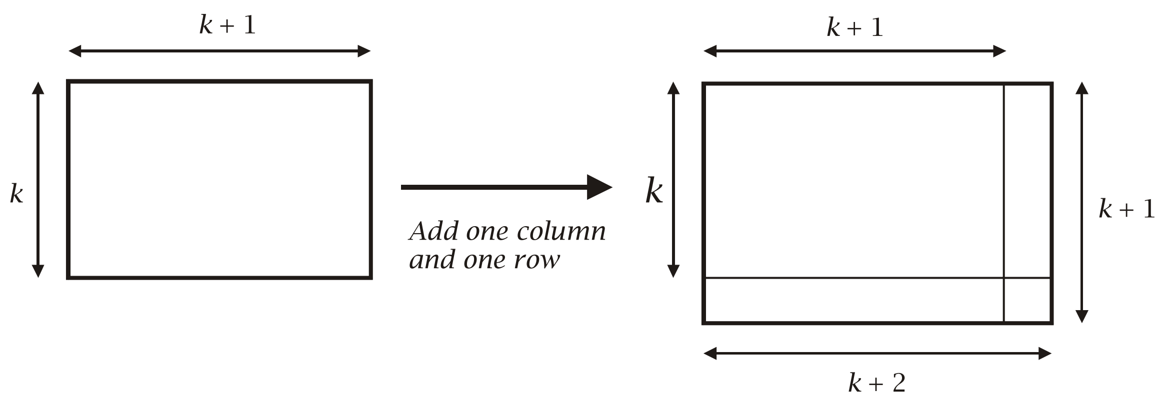
$$S_n = \sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{1}{2}n(n+1)$$

Towards proving a conjecture

The process of constructing triangular numbers might suggest that we have proven this conjecture – for surely we just see that this process of constructing a rectangle out of two triangles could be carried on for ever – and hence applies to any triangular number? Mathematical induction is a process that formalises this argument. However, let us dwell for a moment on why we have *not* strictly proven the result. This is because we have so far examined only *one* particular triangle – the triangle corresponding to the 4th triangular number. We have not actually examined every triangle and consequently we do not know that the result is true for every triangle. Now we proceed to express more explicitly the intuition upon which is based our gut feeling that we have proven the result. This intuition comes from the insight that we could construct a triangle for any number, k; copy it, join the two triangles together and form a $k(k+1)$ rectangle. For example, from the rectangle for $k = 4$, we can construct the rectangle for $k = 5$.



This is done by adding 5 dots to each triangle. Here for $k=4$ the rectangle was of size 4×5 or that is $k(k+1)$. After adding 5 dots to each rectangle, or $k+1$ dots, we obtained the rectangle for $k=5$, which is of size 5×6 or $(k+1)(k+2)$. Writing this all in terms of k, for the kth rectangle the size is $k(k+1)$, then by adding $k+1$ to each side we obtain the k+1th rectangle of size $(k+1)(k+2)$.



If the size of the kth rectangle is $k(k+1)$ then the size of the k+1th rectangle is $(k+1)(k+2)$.

This type of argument is expressed in mathematical induction as the **induction step**. It is an argument to show that *assuming* the result is true for the kth number then the result is true for the (k +1)th number. The induction statement is a *conditional statement* – it says: "If___ then___." Another example of a conditional statement is the sentence, "If I have my umbrella then it is raining" But, if true, this does not prove that I have my umbrella; nor does it prove that it is raining. I need *two* statements, in such a case, to prove that it is raining.

If I have my umbrella then it is raining.
I have my umbrella.
Therefore, it is raining.

Formally

If A then B

$\underline{A \qquad\qquad}$

B

The line here is the way of marking the process of drawing an inference, and is equivalent to "therefore", "hence" or "thus".

Proof by mathematical induction

Likewise, proof by **MATHEMATICAL INDUCTION** is a two-step argument.

Induction Step

If the result is true for the *k*th number then the result is true for the *(k+1)*th number.

Particular Result

The result is true for $n = 1$ (or for some other starting value).

From which the inference can be drawn:

Conclusion

The result is true for all n (or for all n greater than the starting value).

We now illustrate this unique form of argument by proving

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$

Note, it is customary to prove the **particular result** first.

Example (1)

Prove $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$

Proof

Step 1: Particular result

Let $n = 1$

Then LHS = $\sum_{i=1}^{1} 1$

RHS= $\frac{1}{2}(1)(1+1)=\frac{1}{2}(1)(2)=1$

LHS = RHS

Hence, the result holds for $n = 1$

Step 2: Induction Step

Assuming the result holds for $n = k$.

i.e. supposing $\sum_{i=1}^{k} i = \frac{1}{2}k(k+1)$

To show that the result holds for $n = k + 1$

$$\sum_{i=1}^{k+1} i = \frac{1}{2}(k+1)(k+2)$$

Now

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \qquad (*) \\ &= (k+1)\left\{\frac{1}{2}k+1\right\} \\ &= \frac{1}{2}(k+1)(k+2)\end{aligned}$$

The step marked (*) is the crucial one in the proof. At that step we have replaced $\sum_{i=1}^{k} i$ using the induction hypothesis $\sum_{i=1}^{k} i = \frac{1}{2}k(k+1)$.

Step 3: Conclusion

The result holds, by mathematical induction, for all n. That is

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$

The general form of the proof is as follows

Step 1: Particular Result

Proof of the particular result by showing LHS = RHS for $n = 1$ (or for some other natural number).

Step 2: Induction Step

Suppose the result is true for $n = k$ and call this the **induction hypothesis**. Then write the result for $n = k + 1$ and split this into two parts. For example

$$\sum_{i=1}^{k+1} P = \sum_{i=1}^{k} P + Q$$

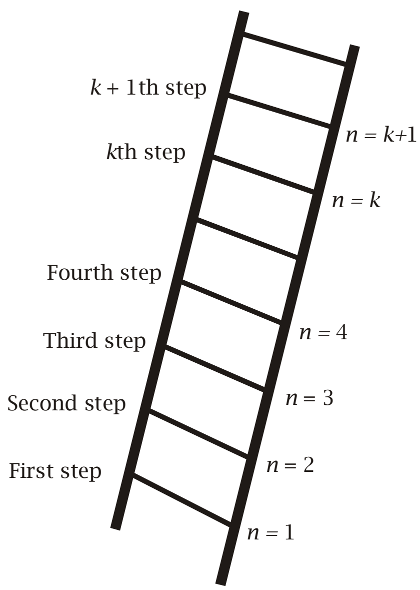
Then substitute the induction hypothesis for P and by doing some algebra (or something) prove the result for $k + 1$.

Step 3: Conclusion

The result holds, by mathematical induction, for all n.

From the finite to the infinite

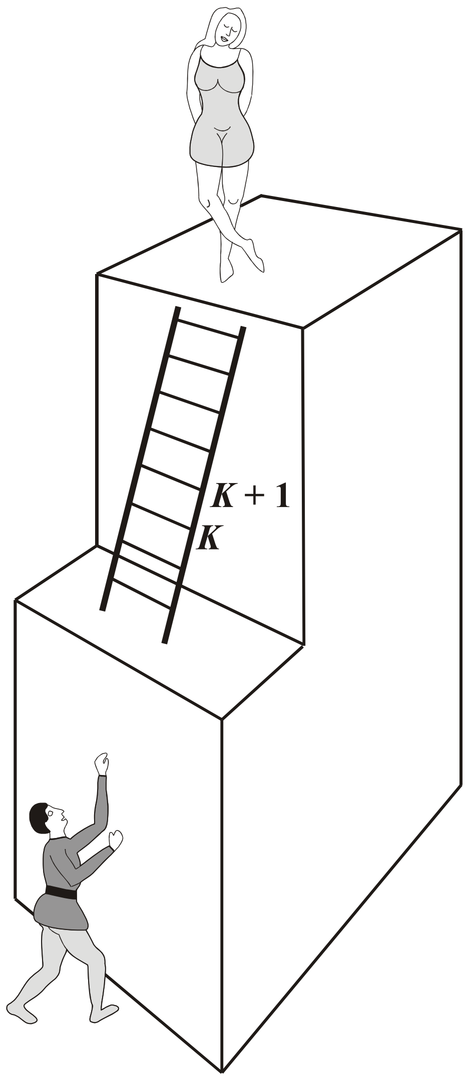
Let us take some time out to explain again why mathematical induction enables one to make conclusions about infinite series from only two starting statements. Mathematical induction has been likened to climbing up a ladder. The steps of the ladder correspond to numbers.



The induction step is equivalent to saying: If I am on the *k*th step then I can climb to the (k +1)th step. Likening mathematical induction to climbing up a ladder illustrates two points.

First Point

In order to know that I can climb the ladder, I must know that I can get onto the ladder in the first place. Look at the cartoon that follows! If Romeo was on the *k*th rung of the ladder, then he could reach the (k + 1)th rung. But he cannot reach the beginning of the ladder. Therefore, he cannot climb the ladder. Therefore, he cannot reach Juliet.



Second Point

If both the particular and the induction steps hold then the MATHEMATICAL INDUCTION entitles us to make a statement that is true for all numbers. In other words mathematical induction enables one to proceed from the PARTICULAR to the INFINITE. We now illustrate the method of mathematical induction with a further example.

Example (2)

Prove: $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

Proof

To prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Particular Step

For $n = 1$: LHS $= \sum_{i=1}^{1} i^2 = 1$

RHS $= \frac{1(1+1)(2\times 1+1)}{6} = \frac{1\times 2\times 3}{6} = 1 =$ LHS

Hence the result holds for $n = 1$.

Induction Step

Suppose $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

This is the induction hypothesis, and we can use this statement at any stage in the proof.

To prove $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6} = \frac{1}{6}(k+1)(k+2)(2k+3)$

It helps to write out what you are trying to prove. This has been obtained from the induction hypothesis by replacing k by $k+1$. Clearly, we have to prove this from the assumption that the induction hypothesis is true and with the help of the rules of algebra.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{[by the induction hypothesis]} \\
&= \frac{(k+1)}{6}\{k(2k+1) + 6(k+1)\} \\
&= \frac{(k+1)}{6}\{2k^2 + k + 6k + 6\} \\
&= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) && \text{[Which is what we wanted to show]}
\end{aligned}$$

Conclusion

The induction step holds and the result is true, by mathematical induction for all *n*.

Mathematical induction and rules for divisibility

We can prove that certain general number formulae are divisible by other numbers using mathematical induction.

Example

Prove that $n^3 + 5n$ is divisible by 6 for all n.

Proof

Let us use the symbol | to mean "divides into". For example 6 | 36 is read "6 divides into 36" or "36 is divisible by 6". [This is standard notation in number theory.]

For the particular result, when $n = 1$

$$n^3 + 5n = 1 + 6 = 6$$
$$6 \mid 6$$

so the result is true for $n = 1$.

For the induction step assume that the result is true for $n = k$. That is

$$6 \mid \left(k^3 + 5k\right)$$

We have to show

$$6 \mid \left((k+1)^3 + 5(k+1)\right)$$

Now

$$\begin{aligned}(k+1)^3 + 5(k+1) &= k^3 + 3k^2 + 3k + 1 + 5k + 5 \\ &= k^3 + 5k + 3k^2 + 3k + 6 \\ &= \left(k^3 + 5k\right) + 3\left(k^2 + k + 6\right)\end{aligned}$$

If k is an odd number then $k^2 + k$ is even (as the sum of two odd numbers), then $k^2 + k + 6$ is even and $6 \mid 3\left(k^2 + k + 6\right)$.

If k is an even number then $k^2 + k + 6$ is even, and $6 \mid 3\left(k^2 + k + 6\right)$.

In either case $6 \mid 3\left(k^2 + k + 6\right)$.

By the induction hypothesis $6 \mid \left(k^3 + 5k\right)$, so

$$6 \mid \left\{\left(k^3 + 5k\right) + 3\left(k^2 + k + 6\right)\right\}$$

since 6 divides both halves separately. Thus the result is true for $k + 1$ and the induction step holds.

Hence, by mathematical induction $n^3 + 5n$ is divisible by 6 for all n

Mathematical induction everywhere

Mathematical induction crops up everywhere. It is used, for example, to prove the rule for differentiation

$$\frac{d}{dx}x^n = nx^{n-1}$$

and many other rules. It is also used to prove results about matrices. The Binomial theorem is proven by mathematical induction. As regards mathematical induction, no new theory is required, but as these we have not stated knowledge of the calculus or matrices as prerequisites of this chapter, we reserve their introduction to later chapters.