Matrix representation of the Fibonacci numbers

The Fibonacci sequence

The pen name of Leonardo of Pisa was Fibonacci. Around 1200 AD Fibonacci posed the following problem: how would an isolated colony of rabbits grow from one pair of rabbits, assuming that each adult pair produced one mixed pair of rabbits each month, and each new pair became productive from the second month onwards?

Solution

In the first month there is just one pair of rabbits; this pair is reproductive, and produces a second pair of rabbits.

In the second month, the original pair are reproductive and produce another (third) pair. The second pair does not produce offspring.

In the third month, both the original and second pair are reproductive and produce offspring. The third pair does not produce offspring.

And so forth. The following table indicates the number of reproductive and non-reproductive pairs at each stage.

Month	Reproductive pairs of rabbits	Non-reproductive pairs of rabbits	Total number of pairs of rabbits
1	1	0	1
2	1	1	2
3	2	1	3
4	3	2	5
5	5	3	8
6	8	5	13

At each stage the total number of pairs is the sum of the reproductive and non-reproductive pairs.

The sequence of numbers

1, 2, 3, 5, 8, 13, ….

is called the *Fibonacci sequence*. It is an example of a second-order recurrence relationship, in which the $(n+1)^{\text{th}}$ term is defined in terms of the n^{th} and $(n-1)^{\text{th}}$ terms, thus

$$a_{n+1} = a_n + a_{n-1}$$

where a_n is the n^{th} term. Also we have

$a_0 = 0$ and $a_1 = 1$

This is called a second-order recurrence relationship because the $(n+1)^{\text{th}}$ term is defined by an operation on the two preceding terms.

General formula for the Fibonacci sequence

The problem is to obtain a general formula for the Fibonacci numbers; that is, a formula of the type

$$a_n = f(n)$$

where *f* is a function. To find this formula we use a matrix representation of the Fibonacci numbers. This is given by

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}$$

To verify this, we start with the matrix $\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$\begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Continuing

$$\begin{pmatrix} a_3 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a_4 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

So in terms of the original problem about rabbit populations, the first term in the matrix column is the number of reproductive pairs, and the second term is the number of non-reproductive pairs. So the matrix does represent the Fibonacci sequence.

Let $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then let us use the idea of the decomposition of this matrix into its canonical diagonal form to solve the problem. That is, we will write T in the form

$$T = QDQ^{-1}$$

where D is a diagonal matrix containing eigenvalues along its diagonal, and Q and Q^{1} are inverses of each other.

To find the diagonal matrix and the eigenvalues, we use the formula

$$\det(T - \lambda) = 0$$

where λ is an eigenvalue

That is

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

Hence

$$(1-\lambda)(-\lambda)-1=0$$
$$\lambda^2-\lambda-1=0$$

Applying the quadratic formula to this, we get

$$\lambda=\frac{1\pm\sqrt{1-(-4)}}{2}$$

That is

$$\lambda=\frac{1+\sqrt{5}}{2} \text{ or } \lambda=\frac{1-\sqrt{5}}{2}$$

This gives the diagonal matrix as

$$D=\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

or

$$D=\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$

Now to find the eigenvectors

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix}$$

Hence

$$x + y = \alpha x \qquad (1)$$

$$x = \alpha y \qquad (2)$$

From (2)

$$y = \frac{1}{\alpha} x$$

giving the first eigenvector as

$$\begin{pmatrix} 1 \\ \frac{1}{\alpha} \end{pmatrix}$$

Similarly, the second eigenvector is

$$\begin{pmatrix} 1 \\ \frac{1}{\beta} \end{pmatrix}$$

So the matrix Q is

$$\begin{pmatrix} 1 & 1 \\ \frac{1}{\alpha} & \frac{1}{\beta} \end{pmatrix}$$

To find the inverse of Q we use the formula

$$Q^{-1} = \frac{1}{\det Q} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

That is

$$Q^{-1} = \frac{1}{\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)} \begin{pmatrix} \frac{1}{\beta} & -1 \\ -\frac{1}{\alpha} & 1 \end{pmatrix}$$

$$= \frac{\alpha\beta}{\alpha - \beta} \begin{pmatrix} \frac{1}{\beta} & -1 \\ -\frac{1}{\alpha} & 1 \end{pmatrix}$$

$$= \frac{1}{\alpha - \beta} \begin{pmatrix} \alpha & -1 \\ -\beta & 1 \end{pmatrix}$$

Now

$$\alpha - \beta = \left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right) = \sqrt{5}$$

hence

$$Q^{-1} = \frac{1}{\alpha - \beta} \begin{pmatrix} \alpha & -1 \\ -\beta & 1 \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{\beta}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

and

$$T = QDQ^{-1} = \begin{pmatrix} 1 & 1 \\ \frac{1}{\alpha} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \frac{\alpha}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{\beta}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

To find the n^{th} term of the series we have

$$a_n = T^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

That is

$$\begin{aligned}
\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} &= T^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= QD^nQ^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ \frac{1}{\alpha} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}^n \begin{pmatrix} \frac{\alpha}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{\beta}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ \frac{1}{\alpha} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix} \begin{pmatrix} \frac{\alpha}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{\beta}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ \frac{1}{\alpha} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix} \begin{pmatrix} \frac{\alpha}{\sqrt{5}} \\ -\frac{\beta}{\sqrt{5}} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ \frac{1}{\alpha} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \frac{\alpha^{n+1}}{\sqrt{5}} \\ -\frac{\beta^{n+1}}{\sqrt{5}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\alpha^{n+1}}{\sqrt{5}} - \frac{\beta^{n+1}}{\sqrt{5}} \\ \frac{\alpha^n}{\sqrt{5}} - \frac{\beta^n}{\sqrt{5}} \end{pmatrix}
\end{aligned}$$

Hence

$$a_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right)$$

or

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Since $\beta < 1$ this means that as n becomes very large

$$a_n \to \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2$$

Also the ratio of two successive numbers in the sequence is

$$\frac{a_{n+1}}{a_n} \approx \frac{\frac{1}{\sqrt{5}}\alpha^{n+1}}{\frac{1}{\sqrt{5}}\alpha^{n}} = \alpha = \frac{1+\sqrt{5}}{2} \approx 1.618...$$

This number is called the golden ratio.

Other second order recurrence relations can be solved in the same way by a matrix representation.