Matrix Algebra

Prerequisites

You should be familiar with vectors and know how to write them in row and column form as well as use of the **i, j, k** notation.

$$\underline{\mathbf{r}} = \overrightarrow{PQ} = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\mathbf{k} = (x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Example (1)

Let the vector $\underline{\mathbf{r}}$ be the displacement vector from the point $P = (2,1,-1)$ to $Q = (5,3,2)$. Then

$$\mathbf{r} = \overrightarrow{PQ} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = (3,2,3)$$

Matrices

A row or column vector is an array of numbers. A *matrix* is any rectangular array of numbers. For example

$$\begin{pmatrix} 2 & -1 & 3 \\ -1 & 0 & 4 \end{pmatrix}$$

is a matrix – it has 2 rows and 3 columns. Any array of numbers *n* rows and *m* columns is a matrix. To show that the array is a matrix it is placed inside brackets. We use curved brackets here, but the use of square brackets is also common. The study of matrices is very closely related to the study of vectors. Matrices represent either vectors or transformations of vectors. Matrices can be added, subtracted and multiplied.

Addition and subtraction of matrices.

We add subtract matrices by adding or subtracting their corresponding elements. It follows that it is only possible to add and subtract matrices that have the same form - the same number of rows and columns.

Example (2)

Find

$$\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 6 \\ 5 & -8 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 6 \\ 5 & -8 \end{pmatrix} = \begin{pmatrix} 4+(-1) & (-1)+6 \\ 3+5 & 2+(-6) \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 8 & -6 \end{pmatrix}$$

It is not possible to add

$$\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} \text{ to } \begin{pmatrix} -1 & 6 \\ 3 & 0 \\ 4 & 7 \end{pmatrix}$$

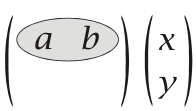
because they have different forms

Multiplication of matrices

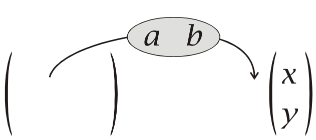
We will learn first the multiplication of a square 2 x 2 matrix by a 2 x 1 column matrix that represents a vector.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

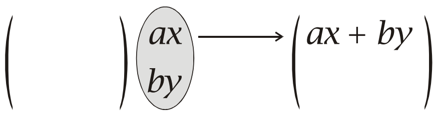
Let us try to visualise this process. The row $\begin{pmatrix} a & b \end{pmatrix}$ is going to be multiplied by the column $\begin{pmatrix} x \\ y \end{pmatrix}$



Imagine lifting out the row $\begin{pmatrix} a & b \end{pmatrix}$ and slotting it besides the column $\begin{pmatrix} x \\ y \end{pmatrix}$



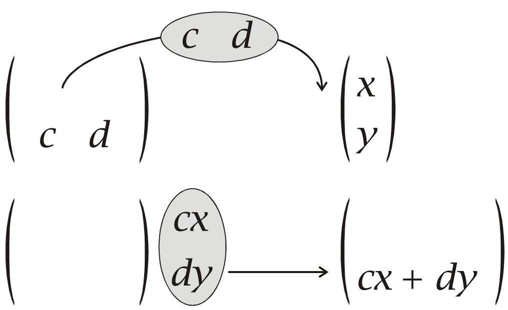
Multiply the corresponding entries, add them together and place the result in the first entry of the resultant column vector.



The overall result is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ \\ \end{pmatrix}$$

Repeat the process for the second row.



The result of the second row goes in the second entry of the column vector.

Example (3)

Find

$$\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} (4 \times 1) & + & (-1 \times -2) \\ (3 \times 1) & + & (2 \times -2) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

This process multiplication of matrices can be generalised to matrices of other sizes.

Example (4)

Find

$$\begin{pmatrix} 6 & 3 & -1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 6 & 3 & -1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} (6 \times -1) & + & (3 \times 4) & + & (-1 \times 2) \\ (2 \times -1) & + & (1 \times 4) & + & (0 \times 2) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

We can only multiply matrices with vectors of the right form. The vector must have as many rows as the matrix has columns. For example we cannot multiply a 2×3 matrix by a 2×1 vector, because the vector is too small and we run out of terms.

$$\begin{pmatrix} 6 & 3 & -1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Multiplication of square matrices

Consider the multiplication of a 2×2 matrix A by another 2×2 matrix B

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = ?$$

The result will be another 2×2 matrix. To find this result, first work out $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix}$ and place the result in the first column.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & \\ cp + dr & \end{pmatrix}$$

Repeat the process for the second row

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} & aq + bs \\ & cq + ds \end{pmatrix}$$

Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

Example (5)

Find $\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -2 & -1 \end{pmatrix}$

Solution

$$\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} (4\times 1)+(-1\times -2) & (4\times 3)+(-1\times -1) \\ (3\times 1)+(2\times -2) & (3\times 3)+(2\times -1) \end{pmatrix} = \begin{pmatrix} 6 & 13 \\ -1 & 7 \end{pmatrix}$$

The multiplication of 3×3 matrices follows the same pattern.

Example (6)

Find

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & -4 & 1 \\ 4 & -3 & 2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & -4 & 1 \\ 4 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 1-6+8 & 0+12-6 & 0-3+4 \\ 0+2+28 & 0-4-21 & 0+1+14 \\ 0+0+4 & 0+0-3 & 0+0+2 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 1 \\ 30 & -25 & 15 \\ 4 & -3 & 2 \end{pmatrix}$$

Matrix multiplication can be generalised to matrices that are not squares.

Example (7)

Find

$$\begin{pmatrix} 6 & 3 & -1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -3 & -1 \\ 2 & 3 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 6 & 3 & -1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -3 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} (6\times 4)+(3\times -3)+(-1\times 2) & (6\times -1)+(3\times -1)+(-1\times 3) \\ (2\times 4)+(1\times -3)+(0\times 2) & (2\times -1)+(1\times -1)+(0\times 3) \end{pmatrix} = \begin{pmatrix} 13 & -12 \\ 5 & -3 \end{pmatrix}$$

The only way to become confident in matrix multiplication is to do plenty of drill right at the beginning. Matrices have many useful applications, but they all require the student to be fluent and confident with matrix multiplication.

Zero and identity matrices

When 0 is added to any number, the result is the same number

$x + 0 = x$

We need the corresponding idea for matrices. The *zero* matrix is any matrix with all zero entries. For example, for 2×2 matrix, the zero matrix is

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Clearly, for any 2×2 matrix A

$A + \mathbf{0} = A$

When any number is multiplied by 1 the result is the same number.

$x \times 1 = x$

The number 1 is called the *identity*. (It is the identity under the operation of multiplication. And 0 is the identity under the operation of addition.). We need an identity also for type of matrix. However, here only square matrices have identity matrices. This is because when a non-square matrix is multiplied by another matrix the result is a matrix of a different size. We saw above, in example (6), that

$$\begin{pmatrix} 6 & 3 & -1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -3 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 13 & -12 \\ 5 & -3 \end{pmatrix}.$$

The identity of a 2×2 matrix is

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The identity of a 3×3 matrix is

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example (8)

Prove for any 2×2 matrix A

$\mathbf{I}A = A$

Solution

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $\mathbf{I} \times A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (1 \times a) + (0 \times c) & (1 \times b) + (0 \times d) \\ (0 \times a) + (1 \times c) & (0 \times b) + (1 \times d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$