Matrix groups

The idea of a matrix group

Matrices are mathematical objects that are generally subject to two matrix binary operations

(1) addition of matrices
(2) multiplication of matrices.

It is, therefore, possible to form groups of matrices whenever one or other of these binary operations is clearly defined, and whenever the resultant objects are of the same form.

For example, the set of all matrices of the form

$$\begin{pmatrix} a & -a \\ -b & a \end{pmatrix}$$

where *a, b* are real numbers forms a matrix group under matrix addition.

The identity is the zero 2×2 matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The inverse of $\begin{pmatrix} a & -a \\ -b & a \end{pmatrix}$ is $\begin{pmatrix} -a & a \\ b & -a \end{pmatrix}$ since

$$\begin{pmatrix} a & -a \\ -b & a \end{pmatrix}+\begin{pmatrix} -a & a \\ b & -a \end{pmatrix}=\begin{pmatrix} a+(-a) & (-a)+a \\ (-b)+b & a+(-a) \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=I$$

The set is also closed under the operation of addition of matrices, since

$$\left\{\begin{pmatrix} a & -a \\ -b & a \end{pmatrix}+\begin{pmatrix} a' & -a' \\ -b' & a' \end{pmatrix}\right\}=\begin{pmatrix} a+(a') & (-a)+(-a') \\ (-b)+(-b') & a+(a') \end{pmatrix}$$

$$=\begin{pmatrix} a+a' & -(a+a') \\ -(b+b') & a+a' \end{pmatrix}=\begin{pmatrix} a'' & -a'' \\ -b'' & a'' \end{pmatrix}$$

is of the same form as

$$\begin{pmatrix} a & -a \\ -b & a \end{pmatrix}$$

Matrix addition is associative.

This verifies that this is a group, and illustrates the formation of matrix groups.

Matrices are intimately connected with transformations of the plane and consequently, corresponding to the transformations of the plane and plane figures there are matrix groups corresponding to the set of linear translations is the set of all matrices of the form

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

where

$a, b \in \mathbb{R}$ under matrix addition.

The set of all rotations about the origin and reflections in lines passing through the origin corresponds to a group of matrices called the general orthogonal group.

To specify the form of matrices of this type we must first define the transpose of a matrix.

The transpose of a matrix A is the matrix $A^{\mathbf{T}}$ obtained when the rows of A are written as columns.

Example

The transpose of $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ is $A^{\mathbf{T}} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Some special matrix groups

The general orthogonal group

The general orthogonal group is the name given to the group of matrices that satisfiy the property

$$AA^{\mathrm{T}} = A^{\mathrm{T}}A = I$$

If it can be shown that the general orthogonal group comprises all rotation and reflection matrices. That is, all matrices of the form

$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and all matrices of the form

$$q_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

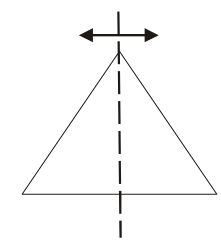
The general rotation group

The general rotation group is a subset of the general orthogonal group and is the group comprising all rotation matrices

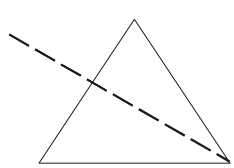
$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Symmetry groups

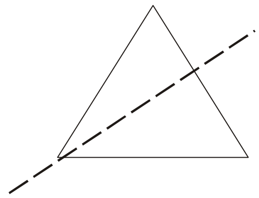
To each symmetry group of a plane figure there corresponds a matrix group. For example, the symmetry group of an equilateral triangle defines a matrix group



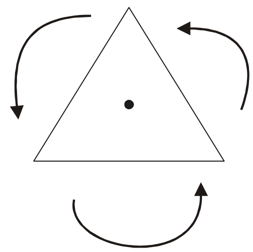
$$q_{\pi/2} = \begin{pmatrix} \cos\pi & \sin\pi \\ \sin\pi & -\cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



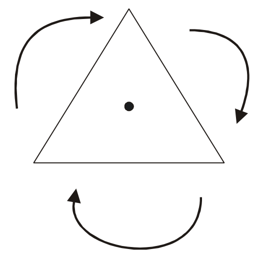
$$q_{-\pi/6} = \begin{pmatrix} \cos\left(-\frac{\pi}{3}\right) & \sin\left(-\frac{\pi}{3}\right) \\ \sin\left(-\frac{\pi}{3}\right) & -\cos\left(\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$



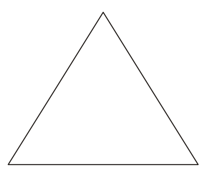
$$q_{\pi/6} = \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & -\cos\left(\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$



$$r_{2\pi/3} = \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & \sin\left(\frac{2\pi}{3}\right) \\ -\sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$



$$r_{-2\pi/3} = \begin{pmatrix} \cos\left(-\frac{2\pi}{3}\right) & \sin\left(-\frac{2\pi}{3}\right) \\ -\sin\left(-\frac{2\pi}{3}\right) & \cos\left(-\frac{2\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$



$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus S_3 is the matrix group

$$\left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

under matrix multiplication