Matrix Transformations

Prerequisites

You should be familiar with matrix multiplication, determinants and the concepts of linear dependence and independence.

Matrices as transformations of the plane

The 2×2 square matrix

In the matrix

$$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

the vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ represent the images of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under M respectively.

Example (1)

Find the image of $(1,0)$ and $(0,1)$ under the matrix $M = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$

Solution

$$\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

You can see that this result does not depend on the specific form of the matrix. That is, $\begin{pmatrix} a \\ b \end{pmatrix}$ is the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ is the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. This means that any 2×2 matrix transforms a parallelogram, *OXYZ*, with one vertex at the origin to another parallelogram. The position of the origin is fixed – that is, it is not moved.

Example (2)

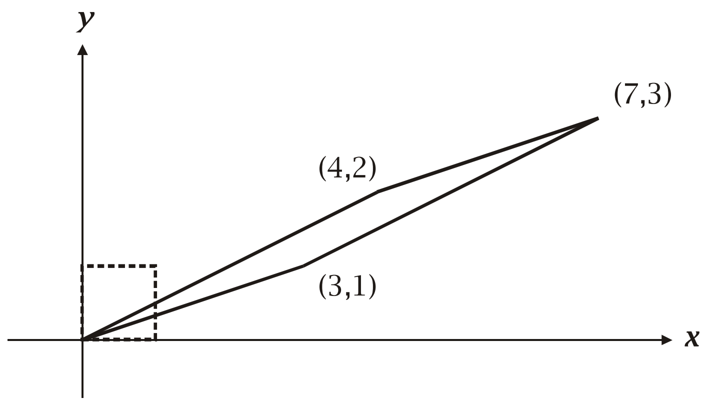
Sketch the image of the square with vertices at the origin, and at $(0,1)$, $(1,0)$ and $(1,1)$ under the matrix $M = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$.

Solution

We have already seen that the image of $(1,0)$ under M is $(4,2)$ and the image of $(0,1)$ under M is $(3,1)$. We can find the image of $(1,1)$ as

$$\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4+3 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

This gives sketch



We can see once again that the image forms a parallelogram regardless of what numbers are substituted in the matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. This is because the image of $(1,1)$ under this matrix is

$$\begin{aligned} \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} a+c \\ b+d \end{pmatrix} \\ &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

So the fourth corner of the parallelogram is always the vector sum of the coordinates of the other two corners (assuming the parallelogram has one corner at the origin). However, we must include the case of the "degenerate" parallelogram – that is, the case were the parallelogram *OXYZ* collapses into a line under the matrix transformation. This occurs with the matrix is not singular.

Example (3)

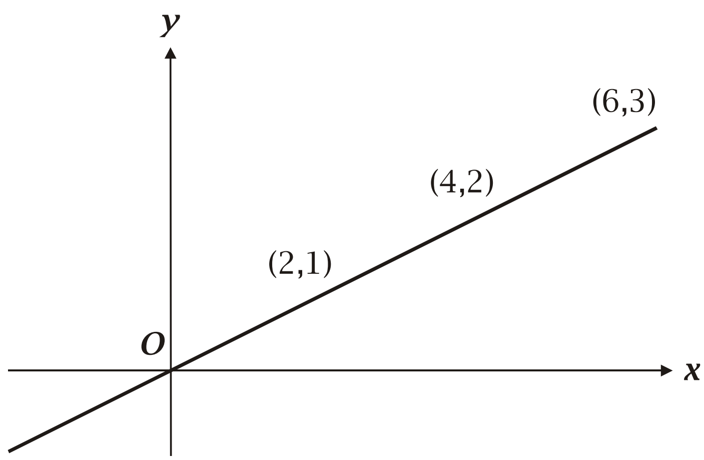
(a) Sketch the image of the square with vertices at the origin, and at $(0,1)$, $(1,0)$ and $(1,1)$ under the matrix $M = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$.

(b) What is the determinant of *M*?

Solution

(a) $\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}$, $\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$, $\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}6\\3\end{pmatrix}$

The points $(4,2)$, $(2,1)$ and $(6,3)$ all lie on the same line $y=\frac{1}{2}x$.



(b) $\det M=\begin{vmatrix}4 & 2\\2 & 1\end{vmatrix}=4-4=0$

So M is a singular matrix.

We know that every singular matrix has row (and column) vectors that are linear combinations of each other, so if M is singular, then the image of $(1,1)$ under M is a linear combination of the images of $(1,0)$ and $(0,1)$. This means that the image of a parallelogram with one vertex at the origin of a singular matrix is a line. Conversely, if the image of a parallelogram collapses into a line under a matrix M then that matrix must be singular. There are some special cases of 2×2 matrices.

Enlargement by a factor k

This has matrix representation

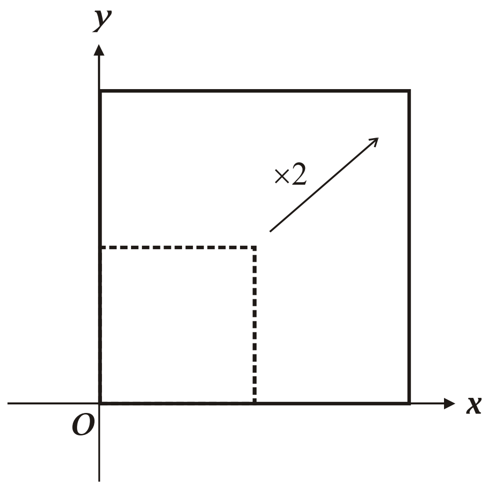
$M=\begin{pmatrix}k & 0\\0 & k\end{pmatrix}$

Example (4)

Sketch the image of the square with vertices at the origin, and at $(0,1)$, $(1,0)$ and $(1,1)$ under the matrix $M=\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$

Solution

$\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$ $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}$ $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}$



Stretches

A stretch parallel to the *x*-axis is represented by the matrix

$M=\begin{pmatrix}k & 0\\0 & 1\end{pmatrix}$

A stretch parallel to the *y*-axis is represented by the matrix

$M=\begin{pmatrix}1 & 0\\0 & k\end{pmatrix}$

Example (5)

Find the matrix corresponding to a stretch of size 4 in the horizontal direction, followed by a stretch of size $\frac{1}{2}$ in the vertical direction. What is the image of the point $(2,-3)$ under the combined transformation?

Solution

This invites us to compose matrices – that is, determine the effect of one matrix is followed by another. Firstly, the horizontal stretch of size 4 has matrix

$M=\begin{pmatrix}4 & 0\\0 & 1\end{pmatrix}$

and the vertical stretch of size $\frac{1}{2}$ has matrix

$$N = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

The combined matrix is

$$NM = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

The expression *NM* means *M* followed by *N*, because we operate on points to the right of the matrix as the image of $(2,-3)$ under this shows

$$\begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ -\frac{3}{2} \end{pmatrix}$$

In this case it does not matter which way around we operate – that is $NM = MN$ for stretches. However, as you should know, matrix multiplication is not usually commutative (as it is in this case) so the order in which transformations take place is important. When the stretch factor is > 1 a stretch is also called a *dilation*.

Shears

A shear parallel to the *x*-axis is represented by the matrix

$$M = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

A shear parallel to the *y*-axis is represented by the matrix

$$M = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

Example (6)

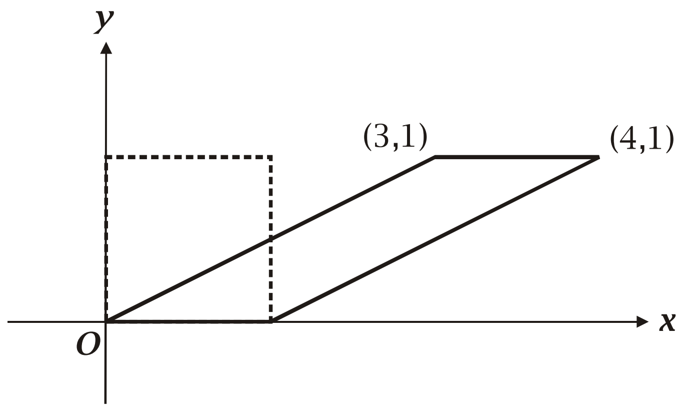
Sketch the image of the square with vertices at the origin, and at $(0,1)$, $(1,0)$ and $(1,1)$ under the shear $M = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

Solution

The image of $(1,0)$ is the first column of *M*, which is $(1,0)$. The image of $(0,1)$ is the second column of *M*, which is $(3,1)$. The point $(1,1)$ maps to

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

and the sketch is



Example (7)

Is a shear parallel to the y–axis of factor k_1 followed by a horizontal stretch of factor k_2 the same as a stretch of factor l followed by a shear of factor k?

Solution

A shear parallel to the y–axis of factor k is represented by the matrix

$$M = \begin{pmatrix} 1 & 0 \\ k_1 & 1 \end{pmatrix}$$

A horizontal stretch of factor l has matrix

$$N = \begin{pmatrix} k_2 & 0 \\ 0 & 1 \end{pmatrix}$$

We are being asked to find out whether these two matrices are commutative; that is, whether $NM = MN$.

$$MN = \begin{pmatrix} 1 & 0 \\ k_1 & 1 \end{pmatrix}\begin{pmatrix} k_2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k_2 & 0 \\ k_1k_2 & 1 \end{pmatrix}$$

$$NM = \begin{pmatrix} k_2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k_1 & 1 \end{pmatrix} = \begin{pmatrix} k_2 & 0 \\ k_1 & 1 \end{pmatrix}$$

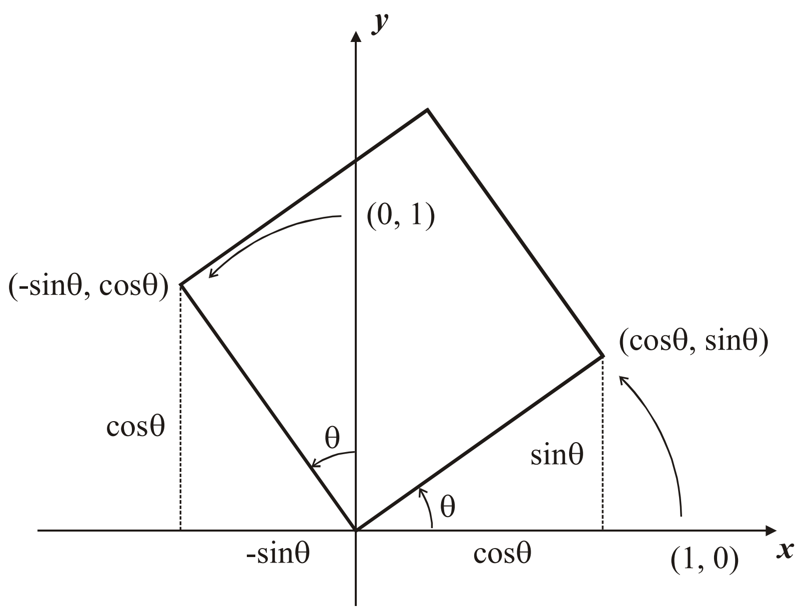
So they are not commutative.

Rotations

A rotation about the origin through the angle θ anticlockwise has a special matrix form. This is

$$R_\vartheta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

To show this we employ trigonometry. Imagine a square with vertices at the origin, and at $(0,1)$, $(1,0)$ and $(1,1)$ rotated about the origin through an angle θ. As the diagram shows, the image of $(1,0)$ under this transformation would be $(\cos\theta, \sin\theta)$, and the image of $(0,1)$ under this transformation would be $(-\sin\theta, \cos\theta)$. Since in the matrix



In the matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ we have $\begin{pmatrix} a \\ b \end{pmatrix}$ as the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ as the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This gives the rotation matrix as

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Example (7)

Write down the matrices corresponding to the following rotations: (a) an anticlockwise rotation through $\pi/2$, (b) a clockwise rotation through $\pi/6$.

Solution

(a) $$R_{\pi/2} = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(b) A clockwise rotation is equivalent to a negative anticlockwise rotation, so we are being asked to find the rotation matrix of $-\pi/6$. To do this recall that sin is and odd function $(\sin(-\theta) = -\sin\theta)$ and cosine is an even function $(\cos(-\theta) = \cos\theta)$.

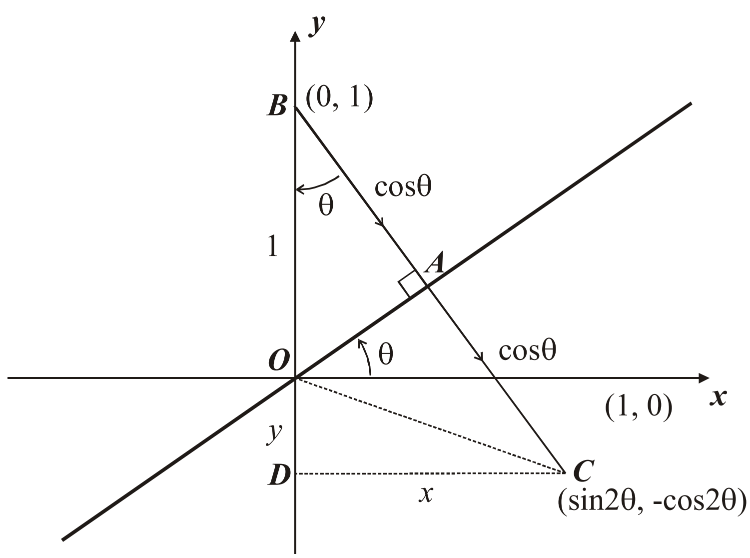
$$R_{-\pi/6} = \begin{pmatrix} \cos(-\pi/6) & -\sin(-\pi/6) \\ \sin(-\pi/6) & \cos(-\pi/6) \end{pmatrix} = \begin{pmatrix} \cos(\pi/6) & \sin(\pi/6) \\ -\sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Reflections

The reflection in the line with angle θ has matrix representation

$$Q_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

To show this we must again apply geometric intuition and use some trigonometry.



In this diagram the line represents the line of reflection with angle θ to the *x*-axis. The arrows represent the process of reflecting the point $(0,1)$ about this line. *OAB* is a right-angled triangle, with hypotenuse of length 1 and angle θ so the length *AB* is $\cos\theta$. *BC* is twice this length, so it is $2\cos\theta$. We need to find the lengths *x* and *y*. In the triangle *BCD* we have

$$\sin\theta = \frac{x}{2\cos\theta}$$

Hence $x = 2\cos\theta\sin\theta = \sin 2\theta$

This being an application of the trigonometric identity, $2\cos\theta\sin\theta = \sin 2\theta$. Also in the triangle *BCD* we have

$$\cos\theta = \frac{1+y}{2\cos\theta}$$
$$y = 2\cos^2\theta - 1 = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

This also requires knowledge of trigonometric identities. This value of y gives the size of the y–coordinate of the point C, but this point lies below the x axis, and hence the coordinates of C are

$$C = \left(\sin 2\theta, -\cos 2\theta\right)$$

That is the image of $(0,1)$ under the reflection through the line with angle θ is $\left(\sin 2\theta, -\cos 2\theta\right)$. By a similar argument we can show that the image of $(1,0)$ under this reflection is $\left(\cos 2\theta, \sin 2\theta\right)$, so the matrix representing this transformation is

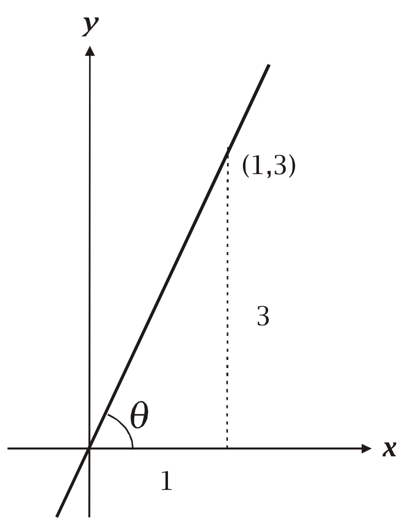
$$Q_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Example (8)

Find the matrix corresponding to a reflection in the line $y = 3x$.

Solution

Let us sketch the line $y = 3x$



From the sketch we see that the angle θ is such that

$$\sin\theta = \frac{3}{\sqrt{10}}$$
$$\cos\theta = \frac{1}{\sqrt{10}}$$
$$\sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{3}{\sqrt{10}}\times\frac{1}{\sqrt{10}} = \frac{3}{5}$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{1}{10} - \frac{9}{10} = -\frac{4}{5}$$

The matrix representing this matrix is therefore

$$Q_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

Combinations of rotations and reflections

We can show that a rotation followed by a rotation is a reflection; a reflection followed by a reflection is a rotation, and that a rotation followed by a rotation is a reflection. We can represent this relationship as follows

	Rotation	Reflection
Rotation	Rotation	Reflection
Reflection	Reflection	Rotation

Example (9)

Find the image of $(1,0)$ (a) under a rotation through $\pi/2$ followed by a reflection in the line $y = 3x$, and (b) the other way around. Sketch the effect of both on the point $(1,0)$ on a single diagram.

Solution

We saw above that the matrix representing a rotation through $\pi/2$ was

$$R_{\pi/2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

And a reflection in the line $y = 3x$ had matrix representation

$$Q_\theta = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

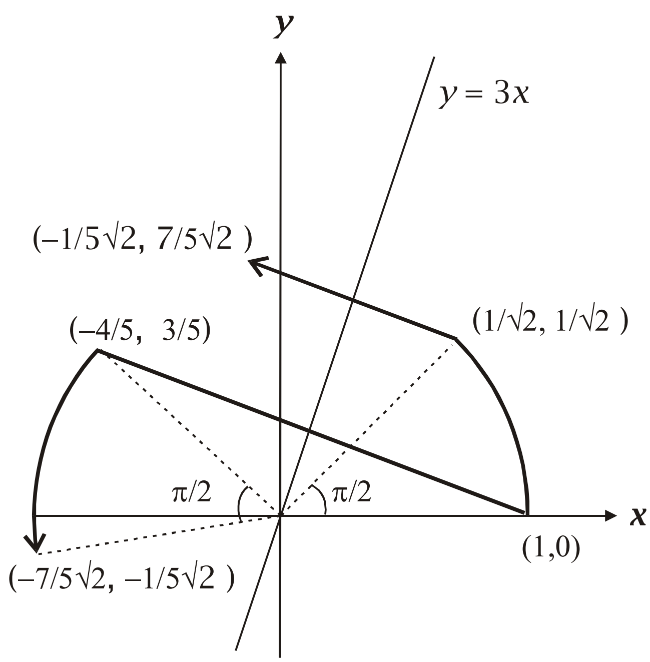
(a) Rotation followed by reflection; this is

$$Q_\theta \circ R_{\pi/2} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{5\sqrt{2}} \\ \frac{7}{5\sqrt{2}} \end{pmatrix}$$

(b) Reflection followed by rotation

$$R_{\pi/2} \circ Q_\theta = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix} = \begin{pmatrix} -\frac{7}{5\sqrt{2}} \\ -\frac{1}{5\sqrt{2}} \end{pmatrix}$$

And the sketch is



Note, in this question we use the small circle $\circ$ to represent the combination of one transformation by another. This combination is called *composition* of transformations (or composition of matrices representing them). It is sometimes convenient to have a symbol to show this, though on other occasions, when there is no ambiguity or confusion, the symbol is left out.

Translations

The transformations considered so far are of the type

$$\phi(x,y)=(ax+by,cx+dy)=\begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

where ϕ is a transformation of the plane (which is the same as a mapping from $\mathbb{R}^2\to\mathbb{R}^2$) and *a, b, c, d* are fixed real numbers. One point about these transformations is that every one of them leaves the origin *fixed* – they do not move the origin. In two dimensions a translation is normally represented by the addition of a fixed vector $\begin{pmatrix}a\\ b\end{pmatrix}$ to the point $\begin{pmatrix}x\\ y\end{pmatrix}$.

$$\begin{pmatrix}x\\ y\end{pmatrix}\mapsto\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}x+a\\ y+b\end{pmatrix}$$

This moves every point of the plain along by *a* and up by *b*. Translations do not leave the origin fixed.

Example (10)

Find the image of (x,y) after a translation by $(2,-1)$ and a clockwise rotation through $\pi/6$.

Solution

The translation is given by

$$t:\begin{pmatrix}x\\ y\end{pmatrix}\mapsto\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}2\\ -1\end{pmatrix}=\begin{pmatrix}x+2\\ y-1\end{pmatrix}$$

The clockwise rotation was found above to be

$$R_{-\pi/6}=\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$$

So

$$R_{-\pi/6}\circ t=\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}x+2\\ y-1\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}(x+2)+\frac{1}{2}(y-1)\\ -\frac{1}{2}(x+2)+\frac{\sqrt{3}}{2}(y-1)\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}x+\frac{1}{2}y+\sqrt{3}-\frac{1}{2}\\ -\frac{1}{2}x+\frac{\sqrt{3}}{2}y-1-\sqrt{3}\end{pmatrix}$$

The representation of translations by 3×3 matrices

It would be useful to be able to represent translations by square matrices and operate with transformations using matrix multiplication. We cannot do this using 2×2 matrices because they all leave the origin fixed, whereas a translation moves the origin. However, there is a useful 3×3 representation of a translation, which does allow us to substitute matrix multiplication for matrix addition, so we can represent the process of composition of transformations by the single matrix operation of matrix multiplication.

Let the 2×2 matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

be represented by the 3×3 matrix

$$\begin{pmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let a point in the plane be represented by the column matrix

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Then the translation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \end{pmatrix}$$

is represented by the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

Example (11)

Find the single 3×3 matrix representing a translation by $(2,-1)$ followed by a anticlockwise rotation through $\pi/2$. To what point does this matrix move the origin? What is the fixed point of this combined transformation?

Solution

The translation by $(2,-1)$ has 3×3 matrix

$$t:\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}$$

The rotation anticlockwise by $\pi/2$ is represented by

$$R_{\pi/2}=\begin{pmatrix}\cos\pi/2 & -\sin\pi/2 & 0\\ \sin\pi/2 & \cos\pi/2 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Hence the single 3×3 representing the translation followed by the rotation is

$$R_{\pi/2}\circ t=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}0 & -1 & 1\\ 1 & 0 & 2\\ 0 & 0 & 1\end{pmatrix}$$

The origin is represented by the column vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$ and this transformation maps this point to $\begin{pmatrix}1\\2\\1\end{pmatrix}$ which corresponds to the point $(1,2)$ in the plane. Let (x,y) be the fixed point of this transformation. Then

$$\begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}0 & -1 & 1\\ 1 & 0 & 2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix}$$

Uncoupling this gives two simultaneous equations

$$x=-y+1$$
$$y=x+2$$
$$y=(-y+1)+2$$
$$2y=3$$
$$y=3/2$$
$$x=-1/2$$

So the fixed point is $\left(-\frac{1}{2},\frac{3}{2}\right)$

Example (12)

The transformations, T_1, T_2 and T_3 in the plane are defined by:

T_1 : A reflection in the line $x + y = 0$.

T_2 : A translation in which the point (x, y) is transformed to the point $(x+2, y-1)$.

T_3 : A clockwise rotation through $\frac{\pi}{2}$ about the origin.

The single transformation T is equivalent to T_1 followed by T_2 followed by T_3.

(a) Find the 3×3 matrix representing T.

(b) Find the equation of the image under T of the line $y = 2x + 1$.

[WJEC June 2004]

Solution

(a) A reflection in the line $x + y = 0$ maps $(1,0)$ to $(0,-1)$ and $(0,1)$ to $(-1,0)$, so its 2×2 matrix is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Alternatively, the angle of $x + y = 0$ is $-\frac{\pi}{4}$ and on substitution into the general formula for a reflection matrix

$$Q_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = Q_{(-\pi/4)} = \begin{pmatrix} \cos\left(-\pi/2\right) & \sin\left(-\pi/2\right) \\ \sin\left(-\pi/2\right) & -\cos\left(-\pi/2\right) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Its 3×3 matrix is $T_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The 3×3 matrix of T_2 is $T_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ which can be seen by $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+2 \\ y-1 \\ 1 \end{pmatrix}$. The rotation through $-\frac{\pi}{2}$ (minus because it is clockwise) about the origin has matrix

$$R_{-\pi/2} = \begin{pmatrix} \cos\left(-\pi/2\right) & -\sin\left(-\pi/2\right) \\ \sin\left(-\pi/2\right) & \cos\left(-\pi/2\right) \end{pmatrix} == \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ or } T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix T is

$T = T_3 T_2 T_1$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) The line $y = 2x + 1$ has vector equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} t \\ 1 + 2t \end{pmatrix}$$

The matrix T acts on this as

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ 1 + 2t \\ 1 \end{pmatrix} = \begin{pmatrix} -1 - t \\ (1 + 2t) - 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 - t \\ 2t - 1 \\ 1 \end{pmatrix}$$

So it transforms this line to the line

$$\mathbf{r} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

with Cartesian equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\begin{cases} x = -1 - t \\ y = -1 + 2t \end{cases}$$

$$y = -1 + 2(-1 - x)$$

$$y + 2x + 3 = 0$$