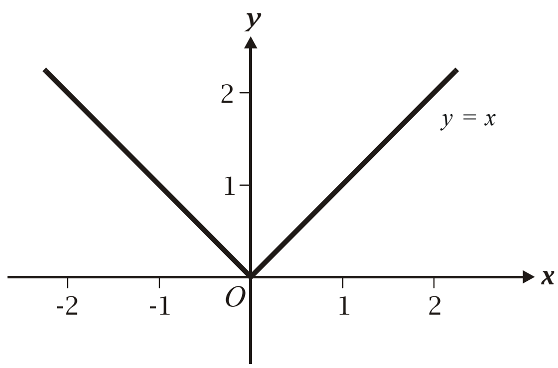
Modulus

Definition of the modulus

The symbol $|x|$ means the size of magnitude of x . It is called the *modulus* of *x*. It is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Its graph is



For x ≥ 0 this is identical to the graph of $y = x$.

For $x < 0$ this is the reflection of $y = x$ in the y-axis.

Example (1)

Solve $3|x| + 3 = 5 - |x|$

Solution

This is a linear equation in $|x|$.

$$3|x| + 3 = 5 - |x|$$
$$4|x| = 2$$
$$|x| = \frac{1}{2}$$
$$x = \frac{1}{2} \quad \text{or} \quad x = -\frac{1}{2}$$

In the following problem we have to remove the modulus sign at the beginning. To do this we use the definition to split the problem into two parts.

Example (2)

Solve $|5x+3|=8$

Solution

$|5x+3|=8$

either $5x+3=8$ or $-(5x+3)=8$

If $5x+3=8$

$5x=5$

$x=1$

If $-(5x+3)=8$

$-5x-3=8$

$-5x=11$

$x=-\frac{11}{5}$

Therefore, either $x=1$ or $x=-\frac{11}{5}$

Modulus inequalities

A modulus inequality is an inequality such as $3x-1>|x-2|$. Such equations are solved by the same method as for modulus equations – by splitting the problem into two parts if necessary. Manipulation of an inequality also obeys the usual rule that if one multiplies by a negative number then one must reverse the sign of the inequality.

Example (3)

Solve $1-3x>|x-2|$

Solution

$1-3x>|x-2|$

Either $x-2>0$ or $x-2<0$

i Suppose $x-2>0$

Then $x>2$ and $1-3x>|x-2|$ implies

$1-3x>x-2$

$-4x>-3$

$x<\frac{3}{4}$

The conditions $x>2$ and $x<\frac{3}{4}$ cannot be simultaneously true.

Therefore it is not possible that $x>2$

ii Let $x-2<0$. Then $x<2$ and $1-3x>|x-2|$ implies

$$1-3x>-(x-2)$$
$$1-3x>-x+2$$
$$-2x>1$$
$$x<-\frac{1}{2}$$

Hence the solution is $x<-\frac{1}{2}$

Intersection of modulus functions

We wish to find the points of intersection of two functions involving a modulus. By sketching the graphs of both functions we are able to deduce equations that do not involve a modulus of the lines (or curves) whose points of intersection are required.

Example (4)

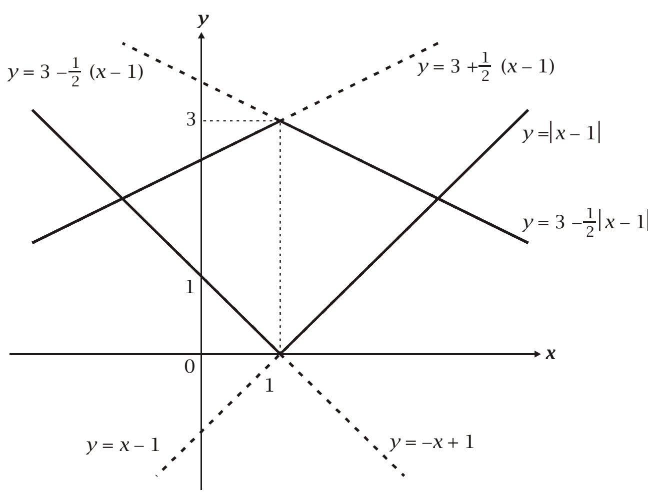
Find the points of intersection of

$y=|x-1|$ and $y=3-\frac{1}{2}|x-1|$

Solution

We begin by sketching the graphs of both functions.

For $y=\|x-1\|$	$x\ge 1$	$\Rightarrow$	$y=x-1$	$m=1$	intercept $=-1$
	$x<1$	$\Rightarrow$	$y=-x+1$	$m=-1$	intercept $=1$
For $y=3-\frac{1}{2}\|x-1\|$	$x\ge 1$	$\Rightarrow$	$y=-\frac{1}{2}x+\frac{7}{2}$	$m=-\frac{1}{2}$	intercept $=\frac{5}{2}$
	$x<1$	$\Rightarrow$	$y=\frac{1}{2}x+\frac{5}{2}$	$m=\frac{1}{2}$	intercept $=\frac{7}{2}$



The graph illustrates that we have to solve

(1) $y = x - 1$ $y = -\frac{1}{2}x + \frac{7}{2}$

(2) $y = -x + 1$ $y = \frac{1}{2}x + \frac{5}{2}$

For (1)

$x - 1 = -\frac{1}{2}x + \frac{7}{2} \Rightarrow x = 3 \quad y = 2$

For (2)

$-x + 1 = \frac{1}{2}x + \frac{5}{2} \Rightarrow x = -1 \quad y = 2$

The points of intersection are $(3,2)$ and $(-1,2)$

Dealing with the modulus in the denominator of a fraction

There are two techniques that can be used to clear the modulus from the denominator of a fraction and so bring it to the top.

(**1**) Use the definition to split the problem into two cases

Example (5)

The definition of the modulus is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Use this definition to solve

$$\frac{|x| + 1}{|x| - 1} < 4$$

Solution

i Suppose $x < -1$ or $x > 1$ so $|x| > 1$

$|x| - 1 > 0$

$|x| + 1 < 4(|x| - 1)$

$|x| + 1 < 4|x| - 4$

$3|x| > 5$

$|x| > 5/3$

$x < -5/3$ or $x > 5/3$

ii Suppose $-1 < x < 1$, so $|x| < 1$

Then $|x| - 1 < 0$

$\therefore |x| + 1 > 4(|x| - 1)$ [Multiplying both sides by a negative number $|x| - 1$ so reverse the sign of the inequality.]

$|x| + 1 > 4|x| - 4$

$3|x| < 5$

$|x| < 5/3$

$-5/3 < x < 5/3$

But we supposed $-1 < x < 1$, so the inequality is satisfied when $-1 < x < 1$

Combining the two inequalities

$x < -5/3$ or $x > 5/3$ or $-1 < x < 1$

(2) Use the property $|x|^2 = x^2$ since the square of any number is always positive.

Example (6)

Use the property $|x|^2 = x^2$ to solve $\frac{|x|+1}{|x|-1} < 4$.

Solution

$$\frac{|x|+1}{|x|-1} < 4$$

$$(|x|+1)(|x|-1) < 4(|x|-1)^2$$

$$x^2 - 1 < 4(x^2 - 2|x| + 1)$$

$$x^2 - 1 < 4x^2 - 8|x| + 4$$

$$8|x| < 3x^2 + 5$$

$$64x^2 < (3x^2 + 5)^2$$

$$64x^2 < 9x^4 + 30x^2 + 25$$

$$9x^4 - 34x^2 + 25 > 0$$

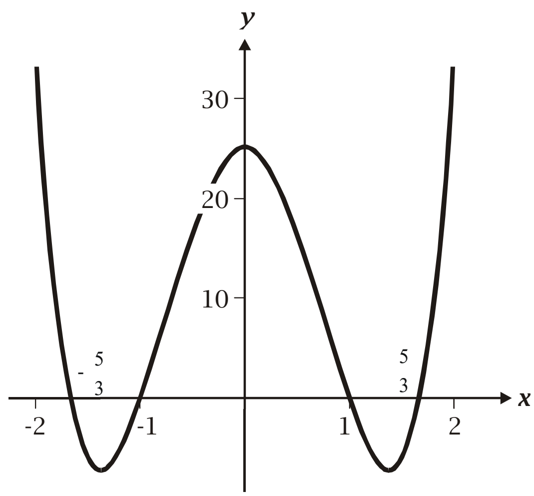
$$\therefore x^2 = \frac{34 \pm \sqrt{34^2 - 4 \times 9 \times 25}}{18}$$

$$x^2 = \frac{34 \pm 16}{18}$$

$$x^2 = 25/9 \text{ or } x^2 = 1$$

$$x = \pm 5/3 \text{ or } x = \pm 1$$

The graph of $y = 9x^4 - 34x^2 + 25$ has roots at $x = \pm 5/3$ or $x = \pm 1$. Between those roots it changes sign and below $x = -5/3$ it is positive. Hence it may be sketched as follows.



$$\frac{|x|+1}{|x|-1} < 4 \text{ when } 9x^4 - 34x^2 + 25 > 0$$

That is when $x < -5/3$ or $x > 5/3$ or $-1 < x < 1$

Comparing the two methods: it is slightly easier to get started with the second method, but as example (6) illustrates the subsequent algebra can get "ugly". It is important to be aware of the property $|x|^2 = x^2$ but we recommend using the first method to solve problems of this type, which relies on the definition of the modulus to split the problem into two sub-problems.