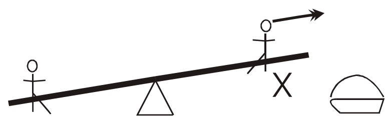
Moment generating functions

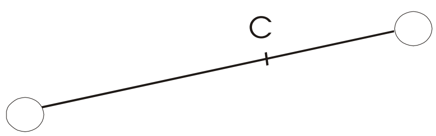
Moments

Moment is originally a concept from Physics used to describe the turning effect of a force.



Imagine two children playing on a see-saw. The turning effect of child X will be greater if either he moves up the bench in the direction marked or he eats a substantial amount from the large picnic hamper and so puts on weight.

When two or more particles are connected in some way they have a joint centre of mass.



The centre of mass, c, will be changed by changing the distance from the centre of mass for one of the particles, or by changing the mass of one of the particles.

From physics

moment = force $\times$ perpendicular distance.

The centre of mass is that point around which all the moments are zero.

These ideas are generalised to random variables. Each value x_i that a random variable X can take together with its probability of occurring

$$P\left(X = x_i\right) = p_i$$

can be thought of as making a contribution to the final location of the mean or central tendency.

The mean is that value of X about which the total moment is 0.

The moment of the value x_i is hence

$$\left(x_i - \mu\right)P\left(X = x_i\right)$$

Where μ is the mean of the distribution. That is, the moment of x_i is the distance of x_i from the mean multiplied by the probability that X takes the value x_i.

The sum of the expressions $\left(x_i - \mu\right)P\left(X = x_i\right)$ could be written:

$$\mu_1 = E\left(X - \mu\right) = \sum_{i=1}^{n}\left(x_i - \mu\right)P\left(X = x_i\right)$$

Since $\mu = E\left(x\right)$ is the central tendency of the distribution it follows that

$$\mu_1 = E\left(X - \mu\right) = 0$$

This derivation of the concept of a first moment about a central value invites generalisation. The expression, $E(X-\mu)^r$, can be given meaning for any integer value of r.

Thus, the rth moment of a random variable X about the mean μ, which is also called the rth central moment, is defined as

$$\mu_r = E\left(X - \mu\right)^r$$

For $r = 0,1,2, \ldots$

For a discrete variable this gives

$$\mu_r = \sum_{i=1}^{n}\left(x_i - \mu\right)^r p\left(X = x_i\right)$$

For a continuous variables this is:

$$\mu_r = \int_{-\infty}^{\infty}\left(x - \mu\right)^r f\left(x\right).dx$$

where $f\left(x\right)$ is the probability density function for X.

Properties of moments

Property 1

$\mu_0 = 1$

Proof

$\mu_0 = E(x-\mu)^0 = E(1) = 1$

Property 2

$\mu_1 = 0$

$\mu_1 = E(x-\mu)^1 = 0$ by definition of μ

Property 3

$\mu_2 = \sigma^2$

That is, the second central moment is the variance.

Proof

$$\mu_2 = E(x-\mu)^2 = \sum_{i=1}^{n}(x_i-\mu)^2 .p(X = x_i)$$

For a discrete distribution, and $\mu_2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x).dx$

In either case, this is the definition of variance.

Property 4

missing

Moment generating functions.

A moment generating function is a function with a Taylor series that has coefficients that enable us to find moments. A moment generating function generates the moments of a random probability distribution. It turns out that moment generating functions have all the same form. Hence, moment generating functions are defined by that form

$$M_x(t) = E(e^{tx})$$

For a discrete variable this gives

$$M_x(t) = \sum_{i=1}^{n} e^{txi} p(X = x_i) = p_1 e^t + p_2 e^{2t} + p_3 e^{3t} + ...$$

And for a continuous variable this gives

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x).dx$$

Where $f(x)$ is the probability density function for X.

Example

A fair six-sided die is thrown once. Let X be the random variable representing the score obtained from this throw. Find the moment generating function of X.

$P(X = n) = \frac{1}{6}$ for $n = 1, 2, 3, 4, 5, 6.$

The probability distribution is:-

n	1	2	3	4	5	6
P(X=n)	1/6	1/6	1/6	1/6	1/6	1/6

Then $M_x(t) = \frac{1}{6}e^t + \frac{1}{6}e^{2t} + \frac{1}{6}e^{3t} + \frac{1}{6}e^{4t} + \frac{1}{6}e^{5t} + \frac{1}{6}e^{6t}$

We will use this example to illustrate further properties of moment generating functions. Firstly, let us calculate E(x) and Var(x) for this distribution by the usual method:

$$E(x) = \sum x_i P(X = x_i)$$
$$= 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{3} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$
$$= \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$$
$$E(x^2) = \sum x_i^{\ 2} P(X = x_i)$$
$$= 1^2\times\frac{1}{6} + 2^2\times\frac{1}{6} + 3^2\times\frac{1}{6} + 4^2\times\frac{1}{6} + 5^2\times\frac{1}{6} + 6^2\times\frac{1}{6}$$
$$= \frac{1+4+9+16+25+36}{6} = \frac{91}{6}$$
$$Var(x) = E(x^2) - \left[E(x)\right]^2$$
$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}$$

We now compute $M_x^{\ 1}(0)$ and $M_x^{\ 2}(0)$:

$$M_x^{\ 1}(0) = \frac{d}{dt}\left(\frac{1}{6}e^t + \frac{1}{6}e^{2t} + \frac{1}{6}e^{3t} + \frac{1}{6}e^{4t} + \frac{1}{6}e^{5t} + \frac{1}{6}e^{6t}\right)\Bigg|_{t=0}$$
$$= \left(\frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t} + \frac{4}{6}e^{4t} + \frac{5}{6}e^{5t} + \frac{6}{6}e^{6t}\right)\Bigg|_{t=0}$$
$$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{7}{2}$$
$$= E(x)$$
$$M_x^{\ 2}(0) = \frac{d}{dt}\left(\frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t} + \frac{4}{6}e^{4t} + \frac{5}{6}e^{5t} + \frac{6}{6}e^{6t}\right)\Bigg|_{t=0}$$
$$= \left(\frac{1}{6}e^t + \frac{2^2}{6}e^{2t} + \frac{3^2}{6}e^{3t} + \frac{4^2}{6}e^{4t} + \frac{5^2}{6}e^{5t} + \frac{6^2}{6}e^{6t}\right)\Bigg|_{t=0}$$
$$= \frac{1}{6} + \frac{2^2}{6} + \frac{3^2}{6} + \frac{4^2}{6} + \frac{5^2}{6} + \frac{6^2}{6}$$
$$= \frac{91}{6}$$
$$= E(x^2)$$

Thus, moment generating functions can be used to find the expectations of $X, X^2, X^3, \ldots$ and, in particular, the variance.

In general

$$E\left(x^r\right)=\frac{d^r}{dt^r}M_x\left(t\right)\Big|_{t=0}$$

We prove the particular cases that

$$E\left(x\right)=\frac{d}{dt}M_x\left(t\right)\Big|_{t=0}$$

and

$$E\left(x^2\right)=\frac{d^2}{dt^2}M_x\left(t\right)\Big|_{t=0}$$

for a discrete variable.

$$\begin{aligned}\frac{d}{dt}M_x\left(t\right)\Big|_{t=0} &= \frac{d}{dt}\left(pe^t+p_2e^{2t}+\ldots+p_ke^{kt}+\ldots\right)\Big|_{t=0}\\ &= p_1e^t+2p_2e^{2t}+\ldots+kp_ke^{kt}+\ldots\big|_{t=o}\\ &= p_1+2p_2+\ldots+kp_k\\ &= \sum_{i=1}^{n}x_ip_i=E\left(X\right)\end{aligned}$$

For a discrete variable, we also have

$$\begin{aligned}\frac{d^2}{dt^2}M_x\left(t\right)\Big|_{t=0} &= \frac{d}{dt}\left(p,e^t+2p_2e^{2t}+\ldots+kp_ke^{kt}+\ldots\right)\big|_{t=0}\\ &= p,e^t+2^2p_2e^{2t}+\ldots+k^2p_ke^{kt}+\ldots\big|_{t=0}\\ &= p_1+2^2p_2+\ldots+k^2p_k\\ &= \sum_{i=1}^{n}x_i^{\;2}.p_i\\ &= E\left(X^2\right)\end{aligned}$$

For continuous variables

$$\frac{d}{dt}M_x(t)\Big|_{t=o} = \left(\frac{d}{dt}\int_{-\infty}^{\infty} e^{tx}.f(x)-dx\right)\Big|_{t=0}$$
$$= \left(\int_{-\infty}^{\infty}\left(\frac{d}{dt}e^{tx}f(x)\right)dx\right)\Big|_{t=0}$$
$$= \left(\int_{-\infty}^{\infty} xe^{tx}f(x).dx\right)\Big|_{t=0}$$
$$= \int_{-\infty}^{\infty} x.f(x)\,dx$$
$$= E(X)$$

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}M_x(t)\Big|_{t=0} = \left(\frac{d}{dt}\int_{-\infty}^{\infty} x.e^{tx}.f(x)\,dx\right)\Big|_{t=0}$$
$$= \left(\int_{-\infty}^{\infty}\frac{d}{dt}x.e^{tx}.f(x).dx\right)\Big|_{t=0}$$
$$= \left(\int_{-\infty}^{\infty} x^2.e^{tx}.f(x).dx\right)\Big|_{t=0}$$
$$= \int_{-\infty}^{\infty} x^2.f(x).dx$$
$$= E(X^2)$$

<u>Example</u>
A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} x/2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the moment generating function, $M_x(t)$ and hence by expanding $M_x(t)$ as a power series in t, find $E(X)$ and $E(X^2)$

$$M_x(t) = \int_0^2 f(x)e^{tx}.dx$$
$$= \int_0^2 \frac{x}{2}e^{tx}.dx = \frac{1}{2}\int_0^2 x.e^{tx}.dx$$

This requires intergration by parts. The intergration by parts formula is

$$\int f.g' = f.g - \int f'g$$

$$f(x) = x \qquad g'(x) = e^{tx}$$

$$f'(x) = 1 \qquad g(x) = \frac{1}{t}e^{tx}$$

Then

$$\begin{aligned}
M_x(t) &= \frac{1}{2}\int_0^2 x.e^{tx}.dx = \frac{1}{2}\left[\frac{x}{t}e^{tx}\right]_0^2 - \frac{1}{2}\int_0^2 \frac{e^{tx}}{t}.dx \\
&= \frac{1}{2}.\frac{2}{t}e^{2t} - \frac{1}{2}\left[\frac{e^{tx}}{t^2}\right]_0^2 \\
&= \frac{e^{2t}}{t} - \frac{1}{2}\left(\frac{e^{2t}}{2t^2} - \frac{1}{2t^2}\right) \\
&= \frac{1}{2\mathrm{t}^2}\left(2te^{2t} - e^{2t} + 1\right) \\
&= \frac{1}{2t^2}\left(e^{2t}(2t-1)+1\right)
\end{aligned}$$

Expanding $M_x(t)$ as a power series

$$\begin{aligned}
M_x(t) &= \frac{1}{2t^2}\left(e^{t2}(2t-1)+1\right) \\
&= \frac{1}{2t^2}\left(\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\ldots\right)(2t-1)+1\right) \\
&= \frac{1}{2t^2}\left(\left(2t+(2t)^2+\frac{(2t)3}{2!}+\frac{(2t)4}{3!}+\ldots\right)-\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\ldots\right)+1\right) \\
&= \frac{1}{2t^2}\left(4t^2-\frac{4t^2}{2}+\frac{8t^3}{2}+\frac{8t^3}{6}+\frac{16t^4}{6}+\frac{16t^4}{24}+\ldots\right) \\
&= \frac{1}{2t^2}\left(2t^2+\frac{8t^3}{3}+2t^4+\ldots\right) \\
&= 1+\frac{4t}{3}+t^2+\ldots
\end{aligned}$$

$$E(X) = M_x'(0) = \frac{d}{dt}\left(1 + \frac{4t}{3} + t^2 + \ldots\right)\Big|_{t=0}$$
$$= \left(\frac{4}{3} + 2t + \ldots\right)\Big|_{t=0}$$
$$= \frac{4}{3}$$
$$E(X^2) = M_x''(0) = \frac{d}{dt}\left(\frac{4}{3} + 2t + \ldots\right)\Big|_{t=0}$$
$$= 2 + \ldots\Big|_{t=0}$$
$$= 2$$

Moment generating function of the sum of independent random variables

The moment generating function of the sum of independent random variables is the product of the moment generating functions for those variables. That is:

$$M_{(x+y)}(t) = M_x(t).M_y(t)$$

Firstly, we illustrate this idea.

Example

[TO BE ADDED]