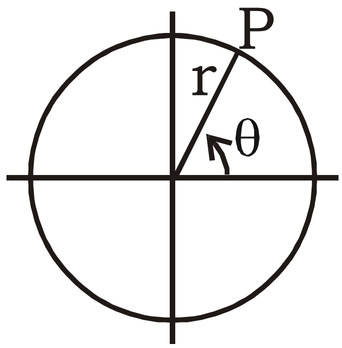
Moment of Inertia

Recall that the position, *P*, of an object on a circular track of radius *r* can be specified by the angle, θ, made with a chosen axis:



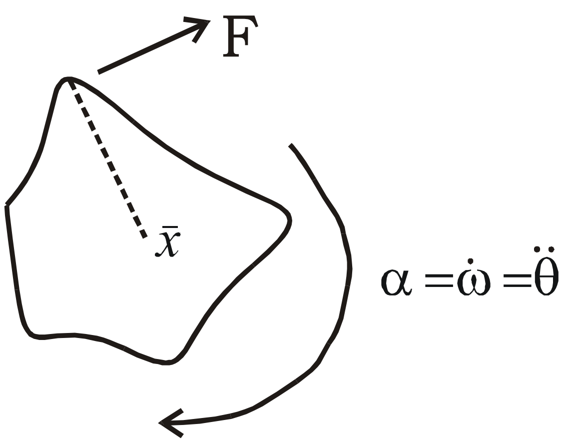
If the object is moving around the track it will have an angular velocity – the rate of change of θ with t.

$$\omega = \theta = \frac{d\theta}{dt}$$

If the object is accelerating around the track it will have an angular acceleration:

$$\alpha = \omega = \frac{d\omega}{dt} = \ddot{\theta} = \frac{d^2\theta}{dt^2}$$

Forces cause objects to accelerate, as indicated by Newton's 2nd law. When a force acts on an object at a distance from its centre of mass it also causes that object to spin. The effect of this is called torque of moment. Just as forces causes objects to accelerate so torques causes angular acceleration – torque causes a change in angular velocity.



Newton's Second Law for angular acceleration states:

Torque $\propto$ angular acceleration

We can compare this with Newton's Second Law;

Force $\propto$ linear acceleration

Then Newton's Second Law leads to the equation:

$F = ma$

Here mass, m, is the constant of proportionality in Newton's Second law. Consequently, mass is more appropriately termed 'inertial mass'.

Designating torque by C and angular acceleration by

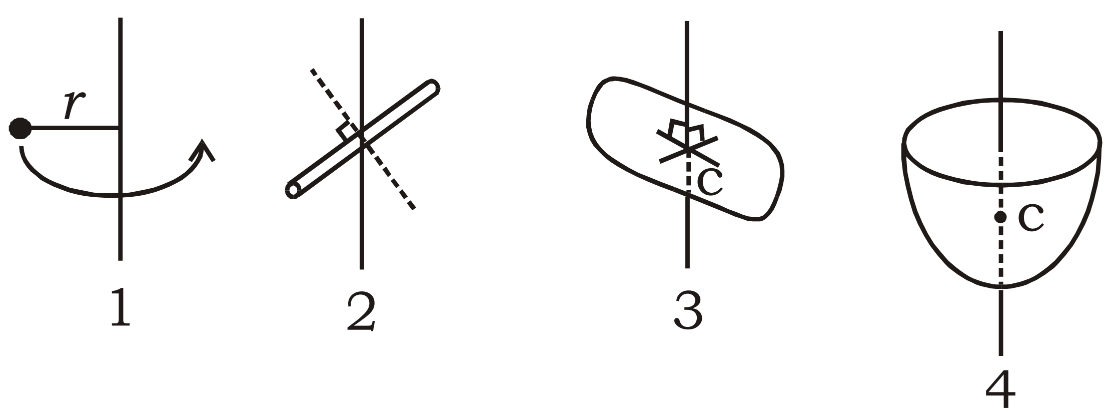
$\alpha = \dot{\omega} = \ddot{\theta}$

we seek an equivalent constant of proportionality for the angular application of Newton's Second Law. The constant, designated *I*, is called the moment of inertia..

Torque = moment of inertia $\times$ angular acceleration

$C = I\ddot{\theta}$ or $C = I\alpha$

Just as inertial mass is a property of every physical body, so moment of inertia is a property if every physical body. The moment inertia must be derived with respect to an axis of rotation. A standard axis will be one that passes through the centre of mass of an object, and, if that object is a rod or a plane lamina, will pass perpendicular to that rod or lamina.



1. A particle is a null dimensional object of mass M but no size. It can have a moment of inertia about any axis at a distance *r*.
2. A rod is a one-dimensional object of theoretically no radius. Any standard axis of rotation passes through the centre of mass c, and is perpendicular to the mass.
3. A lamina is a two-dimensional object of theoretically no breadth. The standard axis of rotation passes through the centre of mass and is perpendicular to the surface.
4. A solid is a three-dimensional object. The standard axis of rotation passes through the centre of mass. However, only solids with an axis of symmetry have an axis of rotation. For asymmetrical solids the moment of inertia is different for each axis of rotation.

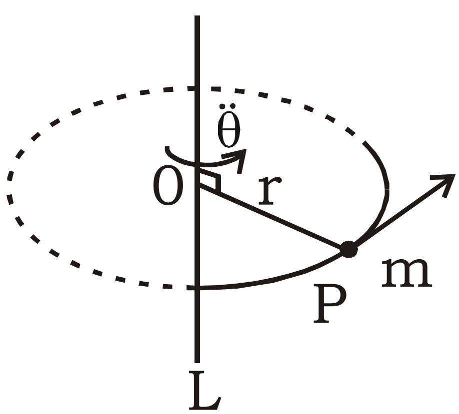
We seek a definition of the moment of inertia it terms of its mass and location of its axis of rotation. We seek rules for the addition of moments of inertia. When two particles are joined together we merely ass their masses to find the inertial mass, but we cannot expect that addition of moments of inertia will obey such simple laws!

Definition of inertia for a particle

Let *P* be a particle of mass *m* rotating about an axis at a distance *r*. Then the moment of inertia *I*, of *P* about this fixed axis is given by

$$I = mr^2$$

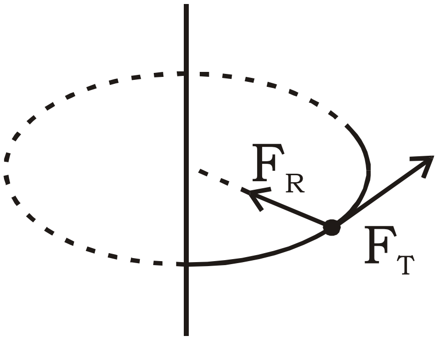
We will now demonstrate the consistency of this definition.



Let *P* be rotating about a fixed axis L passing through a point *O* at a distance *r* from *P*. Let its angular acceleration about this axis be

$$\ddot{\theta}$$

Its linear acceleration may be resolved into two components – one radial, F_R, and the other tangental, F_T, to the orbit.



The linear acceleration of the object can likewise be resolved into radial and tangental components. Radially, the linear acceleration is a centripetal acceleration and is given by:

$$r\omega^2 = r\dot{\theta}^2$$

Tangentally, the linear acceleration is given by:

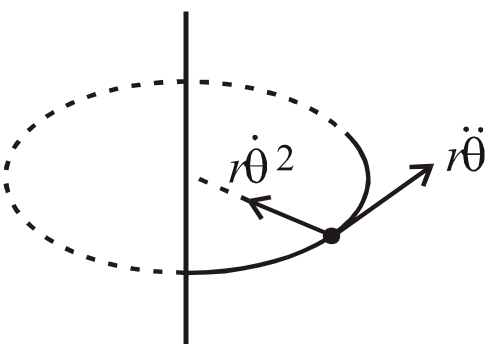
$$\frac{d^2}{dt^2}s = \frac{d}{dt}v = \frac{d}{dt}r\dot{\theta} = r\frac{d^2}{dt^2}\theta = r\ddot{\theta}$$

where

$$s = r\theta$$

is arc length. That is tangental linear acceleration is

$$a_T = r\ddot{\theta}$$



Tangental linear acceleration must obey Newton's Second law, hence:

$$F_T = mr\ddot{\theta}$$

Multiplying both sides by *r*:

$$F_T r = mr^2\ddot{\theta}$$

Bur $F_T r = C$ is the torque (or moment) applied to P. Hence

$$C = I\ddot{\theta}$$

Where $I = mr^2$ is the moment of inertia.

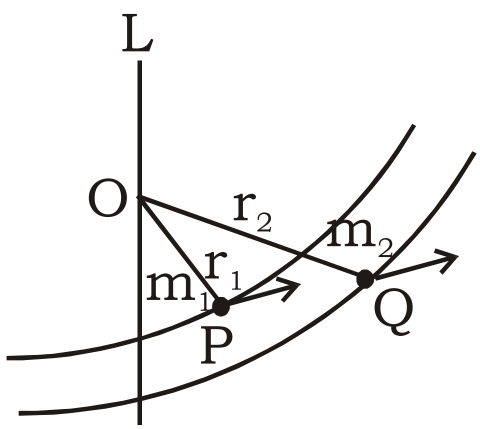
Addition law for moments of inertia for two or more particles rotating about the same axis of rotation

Consider a system of n particles of possibly differing mass M_i and distance r_i (where $1 \le i \le n$) from a fixed axis or rotation, L. If all the particles in the system are rotating together then the moment of inertia of the whole system is given by:

$$I = \sum_{i=1}^{n} M_i r_i^2$$

In other words, for particles we simply add the moments of inertia to each particle to find the moment of inertia of a composite body.

We will demonstrate the validity of this result for a composite body of two particles.



Let P and Q be two particles of mass m_1 and m_2 and perpendicular distance r_1 and r_2 from an axis of rotation L. The total torque applied to this system is:

$$C = C_P + C_Q$$

where

$$C_P = F_P r_1 \quad \text{and} \quad C_Q = F_Q r_2$$

where F_P and F_Q are the tangental components of the forces acting on P and Q respectively.

Both particles are rigidly "fixed" to each other in someway, so their angular acceleration is the same:

$\ddot{\theta}$

The moment of inertia of each particle is:

$$I_P = m_1 r_1^2$$
$$I_Q = m_2 r_2^2$$

Let I = total momentum. Then applying Newton's Second law in the form

$$C = I\ddot{\theta}$$

we have

$$I\ddot{\theta} = I_P\ddot{\theta} + I_Q\ddot{\theta}$$
$$\therefore I = I_P + I_Q = m_1 r_1^2 + m_2 r_2^2$$

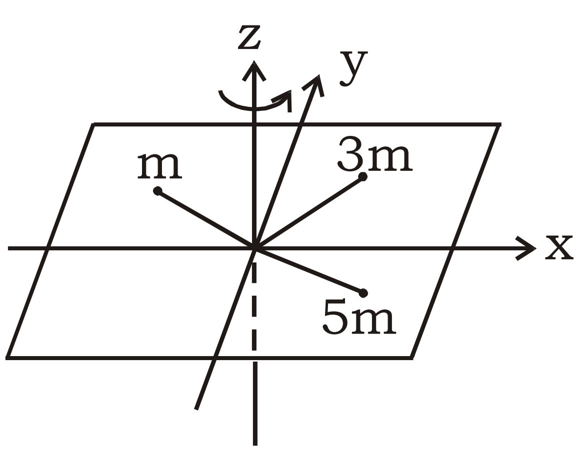
That is $I = \sum_{i=1}^{2} m_i r_i^2$ as required.

The full result for n particles would follow by mathematical induction.

We will now illustrate an application of this result.

Example (1)

Particles of mass m, $3m$ and $5m$ are situated at points (-2, 1), (2, 3) and (4, -1) respectively from O in the x, y plane. Find the moment of inertia of the whole system about the z-axis.



$$r_1 = \sqrt{2^2 + 1^2} = \sqrt{5}$$
$$r_2 = \sqrt{2^2 + 3^2} = \sqrt{10}$$
$$r_3 = \sqrt{4^2 + 1^2} = \sqrt{17}$$

$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ &= m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 \\ &= m \times 5 + 3m \times 10 + 5m \times 17 \\ &= m(5 + 30 + 85) = 120m \end{aligned}$$

Standard results

We now proceed to use the result

$$I = \sum mr^2$$

to derive the standard results for the moment of inertia of a rod.

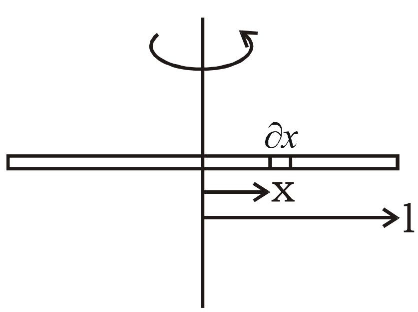
Determination of these standard results by integration from first principals is, however, not required by the syllabus, and these derivations can be omitted. The student may choose to accept the results as standard results and proceed to the next part of the theory.

Rod

Rod of length $2l$ and mass M rotating about an axis perpendicular to the rod through the centre of mass has moment of inertia

$$I = \frac{ml^2}{3}$$

Proof



We divide the rod into segments each of length δx. Let ρ be the density of the rod – that is here the mass per unit length. We treat each element as a particle of mass *M*, given by

$$M = \rho \delta x$$

Each segment is approximately a particle of mass *M*, and hence of moment inertia

$$I_M \approx (\rho \delta x) \times x^2$$

where x is the distance of the segment from the axis of the rotation. Then the moment of inertia of the rod as a whole is given by

$$I \approx \sum I_m$$

And in the limit, as $\delta x \to 0$, the approximation becomes exact:

$$\begin{aligned} I &= \int_{-l}^{l} \rho x^2 dx \\ &= \rho \left[\frac{x^3}{3} \right]_{-l}^{l} \\ &= \rho \left(\frac{l^3}{3} - \left(-\frac{l^3}{3} \right) \right) \\ &= \frac{2\rho l^3}{3} \end{aligned}$$

Now the total mass is given by:

$$M = \sum m = \sum \rho \delta x$$

In the limit, when $\delta x \to 0$

$$\begin{aligned} M &= \int_{-l}^{l} \rho \delta x \\ &= \rho [x]_{-l}^{l} \\ &= 2\rho l \end{aligned}$$

$$\therefore I = \frac{2\rho l^3}{3} = 2\rho l \left(\frac{l^2}{3} \right) = \frac{Ml^2}{3}$$ Shown

The other standard results are proven similarly. The standard results are as follows.

Uniform body of mass M	Axis	Moment of Inertia
Rod of length $2a$	Through the centre and perpendicular to the rod	$\frac{Ma^2}{3}$
Rod	Parallel to the rod and at a distance, d, from it	Md^2
Rectangle of length $2a$ and width $2b$	Passing through the midpoints of the side with length $2a$ and perpendicular to it	$\frac{Ma^2}{3}$
Ring with radius a	Through the centre of the ring and perpendicular to it	Ma^2
Disc with radius a	Through the centre of the disc and perpendicular to it	$\frac{Ma^2}{2}$
Solid sphere with radius a	A diameter	$\frac{2Ma^2}{5}$
Hollow sphere of radius a	A diameter	$\frac{2Ma^2}{3}$
Solid cylinder of radius a	The central axis	$\frac{Ma^2}{2}$
Hollow cylinder of radius a	The central axis	Ma^2