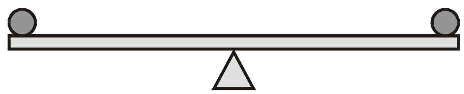
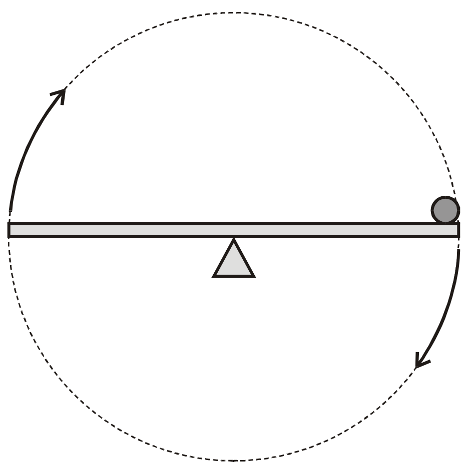
Moments

Pivots

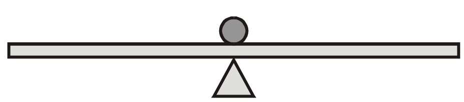
The following diagram shows a beam resting on a pivot placed underneath the beam at its centre. Two balls of equal weight are placed at either end of the beam. The whole system is in equilibrium. The two balls exactly balance each other.



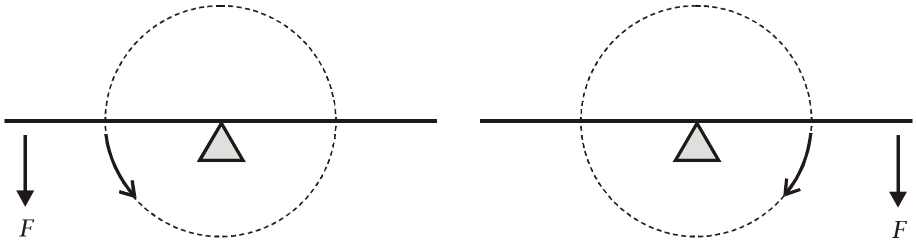
The presence of the pivot means that the beam can *rotate* about the pivot. The beam is not free to fall. If we remove one of the balls, the beam will rotate about the pivot.



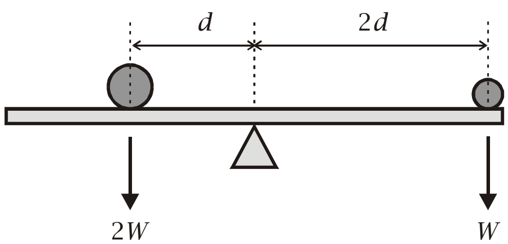
A ball at the centre, just above the pivot, does not produce a turning effect.



Evidently the presence of an object with weight $W = mg$ placed at distance d from the pivot causes a rotational effect. We call this rotational effect *moment* or *torque*. The two terms are synonyms and can be used interchangeably. It is the weight of the object that produces this effect. Weight is an example of a force. Forces produce turning effects when they act around a pivot. The moment may be either *clockwise* or *anti-clockwise*.



The size of the moment depends on the size of the force and the distance of the force from the pivot.



Experiment shows that a force $2W$ at a distance d from the pivot will counter-balance a force W at a distance $2d$ from the pivot.

Moment

The moment of a force or its torque is defined to be

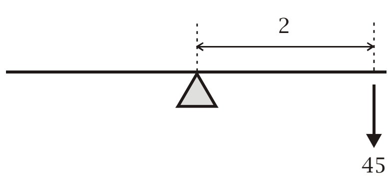
Moment = Force $\times$ perpendicular distance of the force from the axis of rotation

$C = F \times d$

The units of moment are Newton metres (Nm).

Example (1)

The diagram shows a load of weight 45N suspended from the end of a rod 2 m from the pivot. Calculate the moment created by this load.



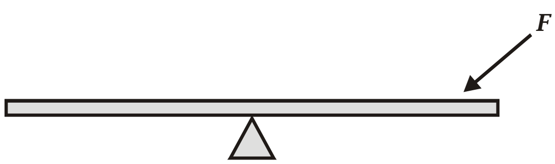
Solution

$$
\begin{aligned}
C &= Fd \\
&= 45 \times 2 \\
&= 90 \text{ Nm}
\end{aligned}
$$

The definition of moment

Moment = Force $\times$ perpendicular distance of the force from the axis of rotation

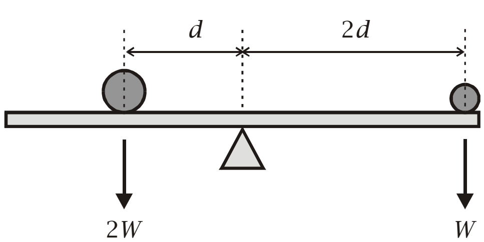
includes the term "perpendicular". This is to deal with situations where the force applied is not already at right angles to the beam.



In the diagram above the force is applied at an angle to the plane of the beam. The moment is not the same as if the force was applied perpendicular to the beam. However, in this chapter we will deal only with examples where the force is already perpendicular.

Equilibrium of non-concurrent forces

When two or more forces act on an object lie in the same plane, they are said to be *coplanar*. When the forces pass through the same point they are said to be *concurrent*. In the case of forces applied at a distance from a pivot then the forces are *non-concurrent*. We have seen that non-concurrent forces produce turning effects. However, these turning effects can cancel out so that the system as a whole is still in equilibrium. We saw already one example of equilibrium of coplanar, non-concurrent forces.



The beam is not rotating and is in equilibrium. The clockwise moment is

$$\begin{aligned} C &= \text{force} \times \text{distance} \\ &= W \times 2d \\ &= 2Wd \end{aligned}$$

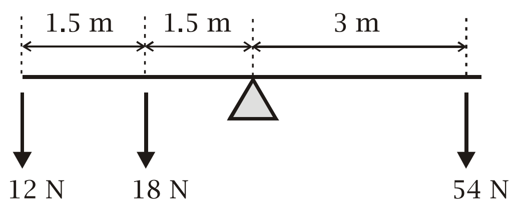
This is the same as the anticlockwise moment

$$\begin{aligned} C &= \text{force} \times \text{distance} \\ &= 2W \times d \\ &= 2Wd \end{aligned}$$

The reason why the system is not rotating is because the clockwise and the anticlockwise moments are the same. This is a general principle.

Example (2)

The diagram shows three forces applied to a uniform rod positioned over a pivot.



Determine whether the system is in equilibrium and if it is not state the magnitude of the resultant moment and in which direction the rod is rotating.

Solution

The force of 54 N produces an anticlockwise moment. This size of this is

$$\begin{aligned} C_{\text{anticlockwise}} &= Fd \\ &= 54 \times 3 \\ &= 162 \text{ Nm} \end{aligned}$$

The other two forces produce clockwise moments. Note that the 12 N force lies at a distance of 3 m from the pivot. The sum of the two clockwise moments is

$$\begin{aligned} C_{\text{clockwise}} &= (12 \times 3) + (18 \times 1.5) \\ &= 63 \text{ Nm} \end{aligned}$$

Since the anticlockwise moment is greater than the clockwise moment the system is not in equilibrium and is rotating anticlockwise. The magnitude of the *resultant* anticlockwise moment is

$$\begin{aligned} C_{\text{resultant}} &= 162 - 63 \\ &= 99 \text{ Nm} \end{aligned}$$

This example illustrates various principles when dealing with a system where more than one moment applies.

(1) Clockwise moments are added to clockwise moments. Anticlockwise moments are added to anticlockwise moments.

(2) The resultant moment is the sum of all the clockwise moments less all the anticlockwise moments or vice-versa.

(3) For an object to be in rotational equilibrium:
the sum of clockwise moments = the sum of anticlockwise moments.

Moments are vectors and hence have a direction. Let us also introduce the convention that a clockwise moment is positive and anticlockwise moment negative. Using this convention, the answer we should write the anticlockwise moment in example (2) is $C_{\text{anticlockwise}} = -162\text{ Nm}$.

Centre of mass

Example (3)

The diagram shows a uniform rod *AB* of mass 3.6 kg resting on a support at *C*. The length of *AB* is 3 m. The distance *AC* is 1 m.

Explain why the rod is not in equilibrium

Solution

In this question the rod is described as being *uniform*. This means that the density of the rod is the same throughout. Up to now we have conveniently ignored the issue of the weight of the rod or beam that rests on the pivot. This was because in those questions the pivot was placed at the centre of the rod or beam. It was tacitly assumed that the beam itself would not cause the system to rotate. In this question we realise intuitively that more of the weight of the beam lies towards *B* than towards *A*, so we expect the beam to be rotating clockwise around the pivot at *C*.

Thus, when the pivot is not in the centre of a uniform rod then the weight of the rod contributes to the moments acting around that pivot. Strictly speaking the weight of a uniform rod is evenly distributed over the whole rod – every part of it has weight. But this makes problem solving complicated, and fortunately there is a simplifying concept also drawn from physics.

The mass of an object is spread out over a certain volume. For instance, the mass of a *uniform* sphere occupies its whole volume evenly. However, when an object is subject to external forces, the object behaves as if its entire mass were concentrated at a single point called its *centre of mass*. So for the purposes of calculating the effect of a force on a sphere, the sphere can be replaced by a single point of the same mass at the centre of mass. This is what is meant by a *particle*. It is a theoretical object of no size and of a given mass that exists at the centre of mass of an object of some definite size and the same mass. The concept of centre of mass has been *implicit* in many questions you may have met – for example, when dealing with uniform acceleration the phrase "a particle accelerates uniformly…" is quite common. Now we have made *explicit* what that means.

The centre of mass of a uniform rod lies at its mid point.

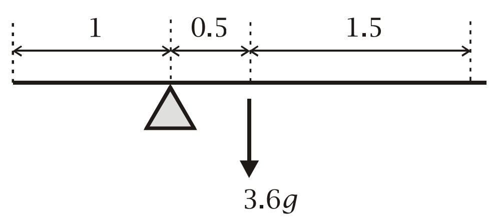
Any force acting on the rod acts as if the entire mass of the rod was located at the rod's centre of mass. A rod balancing on a pivot is subject to gravity and so has weight. The rod's entire weight may be regarded as acting at its centre of mass, which is its mid-point. In the context of weight the centre of mass is called *the centre of gravity*. The entire weight of an object acts as if it was concentrated at its centre of gravity.

Example (3) continued

Calculate the magnitude of the moment acting on the rod.

Solution

We redraw the diagram as follows showing the weight acting at the centre of gravity.



The moment, which is clockwise, has magnitude

$$\begin{aligned} C &= Fd \\ &= 3.6 \times 9.8 \times 0.5 \\ &= 17.64 \\ &= 18 \text{ Nm } (2.\text{s.f.}) \end{aligned}$$

Problem-solving involving non-concurrent forces in equilibrium

The problems we will consider in the remainder of this chapter will be concerned with non-concurrent forces that are in equilibrium. The fact that they are in equilibrium means that we can determine the magnitude of an unknown quantity, as the next example illustrates.

Example (4)

The diagram shows a uniform rod *AB*, with two particles attached to the rod at *A* and *B*, resting horizontally in equilibrium on a smooth support at *C*.

The length of *AB* is 2.4 m and its mass is 0.8 kg. The masses of the particles *A* and *B* are 1.8 kg and *m* kg respectively.

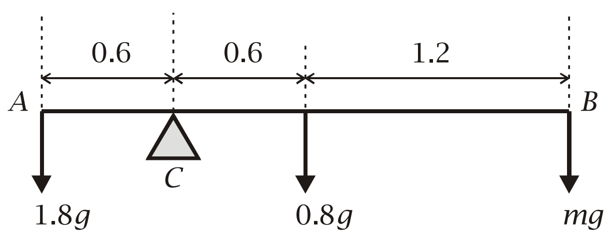
(*a*) Given that the distance *AC* is 0.6 m find *m*.

(*b*) Find the magnitude of the reaction at *C*.

Solution

This question introduces the term *smooth* to describe the support at *C*. This is the usual term to indicate that we can ignore the effect of friction in the question.

(*a*) We redraw the diagram marking in the forces, including the weight of the rod acting at its centre of gravity.



Applying the principle

Sum of anticlockwise moments = Sum of clockwise moments

$1.8g \times 0.6 = (0.8g \times 0.6) + (1.8mg)$

$1.08 = 0.48 + 1.8m$

$m = \frac{0.6}{1.8} = 0.33 \text{ kg } (2.\text{s.f.})$

(*b*) The pivot at *C* is holding up the whole weight of the rod and the two masses. So the reaction at *C* is equal and opposite to all these forces combined.

$$\begin{aligned} R &= 1.8g + 0.8g + 0.33g \\ &= 28.714 \\ &= 29 \text{ N } (2.\text{s.f.}) \end{aligned}$$

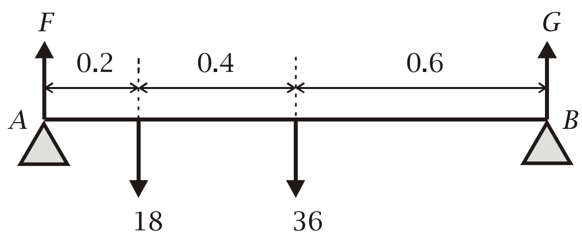
A rod may rest in equilibrium on two pivots. The next example considers a problem of this type.

Example (5)

A uniform rod *AB* of length 1.2 m and weight 36 N rests horizontally on smooth supports at *A* and *B*. A load of 18 N is attached to the rod at a distance 0.2 m from *A*. Find the reactions at *A* and *B*.

Solution

Let the reactions at *A* and *B* be *F* and *G* respectively. The following diagram shows the forces acting on the rod.



The rod is in equilibrium, meaning that it is not rotating about either pivot *A* or *B*. That means that the sum of the moments around either must be zero. So we can *take moments about A* and we can *take moments about B*. When we take moments about *A* the reaction *F* will contribute zero moment, so we will obtain an equation only involving *G*. Likewise, when we take moments about *B* then *G* will contribute zero moment, so we will obtain an equation only in *F*.

Taking moments at *A*

Sum of clockwise moments = Sum of anticlockwise moments

$$(18 \times 0.2) + (36 \times 0.6) = 1.2G$$

$$G = \frac{25.2}{1.2} = 21 \text{ N}$$

Taking moments at *B*

Sum of clockwise moments = Sum of anticlockwise moments

$1.2F = (18 \times 1.0) + (36 \times 0.6)$

$F = \dfrac{39.6}{1.2} = 33 \text{ N}$

It is possible to check the answers in this problem. Since the whole system is in static equilibrium the sum of the reactions at the pivots must equal sum of the weights. The reactions are

$F + G = 33 + 21 = 54 \text{ N}$

The weights are

$\text{weight of rod + load} = 36 + 18 = 54 \text{ N}$

Since the two agree this confirms that the reactions at *A* and *B* are 33 N and 21 N respectively.

We have already remarked that the units of moment are Newton metres (Nm). Physicists give standard units to every physical quantity and these are the standard units for moment. (These are the *System International*, SI, units of moment.) However, in questions involving equilibrium we use the principle

Sum of clockwise moments = Sum of anticlockwise moments

This means that the units cancel out on both sides. This means that it is not necessary to convert distances given in, say, centimetres into distances given in metres. Provided all the distances are expressed in the same units the answer will be the same. In the next example, the distances are given in cm.

Example (6)

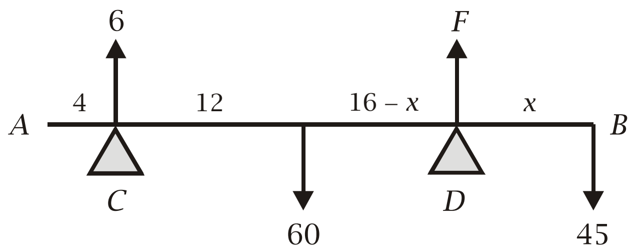
The diagram shows a uniform rod *AB* of length 32 cm and weight 60 N resting horizontally on two smooth supports *C* and *D* where *AC* = 4 cm and *BD* = *x* cm. A load of 45 N is placed at *B*.

(*a*) Given that the reaction at *C* is 6 N, find the reaction at *D* and *x*.

(*b*) Find the greatest value of *x* for which the rod remains in equilibrium.

Solution

(*a*) Let the reaction at *D* be *F*.



The value of *F* must be sufficient to support the weight of the rod and the load less the reaction at *C*, which is 6 N. So

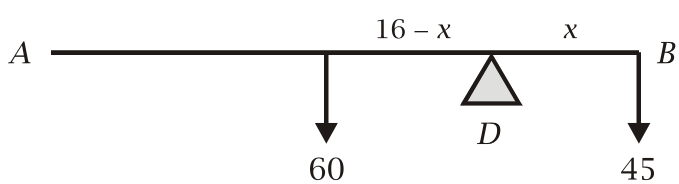
$$6 + F = 60 + 45$$
$$F = 99 \text{ N}$$

To find *x* we can take moments at either *C* or *D*. Taking moments at *C*.

Sum of clockwise moments = Sum of anticlockwise moments

$$(60 \times 12) + (45 \times 28) = 99 \times (28 - x)$$
$$\frac{1980}{99} = 28 - x$$
$$x = 8 \text{ cm}$$

(*b*) As *x* increases the reaction at *C* decreases, until it is zero. At this point the rod is just balancing on the support at *D*, so in effect the support at *C* ceases to make a contribution. We can redraw the diagram as follows.



Taking moments at *D*.

Sum of clockwise moments = Sum of anticlockwise moments

$$45x = 60(16 - x)$$
$$105x = 960$$
$$x = 9.142\ldots$$
$$x = 9.1 \text{ cm } (2\text{.s.f.})$$