Motion in a Vertical Circle

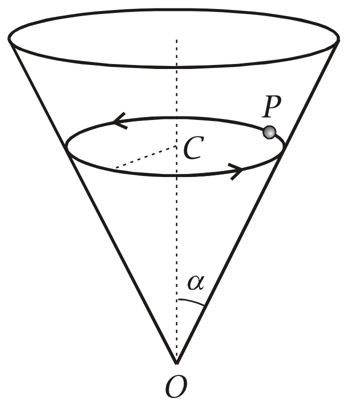
Prerequisites

You should be familiar with (1) motion in a horizontal circle and (2) energy conversions between gravitational and kinetic energy.

Summary of motion in a horizontal circle

A particle moves in a horizontal circle of radius r with constant speed v when it is subject to a single resultant centripetal force directed from the centre of mass of the particle to the centre of the circle such that centripetal force is given by $F = \frac{mv^2}{r} = mr\omega^2$ where $\omega = \text{angular velocity}$. The centripetal acceleration is $a = \frac{v^2}{r} = r\omega^2$. The relationship between the angular velocity (ω) and the speed (v) is $v = r\omega$.

Example (1)

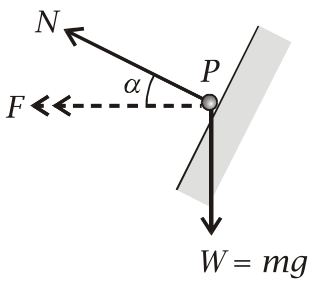


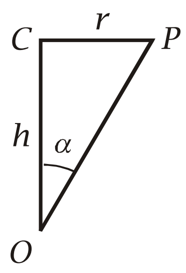
The diagram shows a fixed hollow smooth cone with semi-vertical angle α and vertex *O*. A particle *P* of mass m kg is describing a horizontal circle with centre *C* with constant speed 2.8 ms^{-1} on the inner surface of the cone.

(*a*) Find the height of *C* above *O*.

(*b*) Given that the angular speed is 1.4 rad s^{-1} find α.

Solution





(*a*) Let the height of *C* above *O* be *h*. From trigonometry we also have

$$\tan\alpha = \frac{r}{h}$$

There are two forces acting on the particle *P*. These are the weight $(W = mg)$ of the particle, and the normal reaction (N). The resultant of these two forces is the centripetal force $\left(F = \frac{mv^2}{r}\right)$. Resolving horizontally and vertically

$$(\rightarrow) \qquad N\cos\alpha = \frac{mv^2}{r}$$

$$(\uparrow) \qquad N\sin\alpha = mg$$

Hence

$$\tan\alpha = \frac{mg}{\left(\frac{mv^2}{r}\right)} = \frac{rg}{v^2}$$

$$\frac{r}{h} = \frac{rg}{v^2}$$

$$h = \frac{v^2}{g} = \frac{(2.8)^2}{9.8} = 0.8 \text{ m}$$

(*b*) From $v = r\omega$

$$2.8 = r \times 1.4$$

$$r = \frac{2.1}{1.5} = 2$$

$$\tan\alpha = \frac{r}{h} = \frac{2}{0.8} = 2.5$$

$$\alpha = 68.2^\circ \ (0.1^\circ)$$

In practice such a particle in example (1) would lose energy owing to friction between the particle and the surface of the cone. This loss of energy would result in the particle slowing down, so a constant speed of v ms^{-1} could not be maintained.

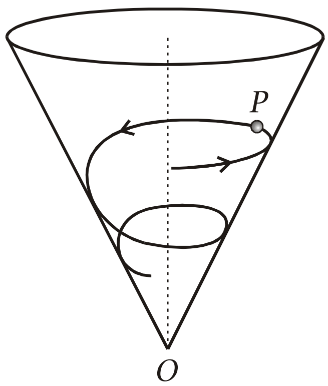
Example (1) continued

(*c*) Explain what would happen to the trajectory of particle *P* in example (1) as it slows down.

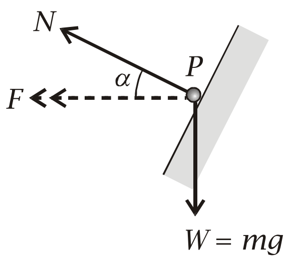
(*d*) What happens to the magnitude of the centripetal force as the particle *P* slows down?

Solution

(*c*) It would spiral down.



(*b*) The centripetal force would remain the same throughout the motion of the particle, even as it spirals down. This is because the centripetal force is provided by the horizontal component of the normal reaction at the interior surface of the cone. This in turn is determined by the weight (mass) of the particle. None of these vary so the centripetal force remains the same.



To explain this further the centripetal force is given by

$$\text{centripetal force} = \frac{mv^2}{r}$$

If the centripetal force remains constant, then as v decreases so must r. Hence the particle spirals down.

Example (2)

Consider the equation

$$\text{centripetal force} = \frac{mv^2}{r}.$$

Suppose in this equation the velocity of a particle v is increasing but the radius r remains constant. What can you say about the centripetal force?

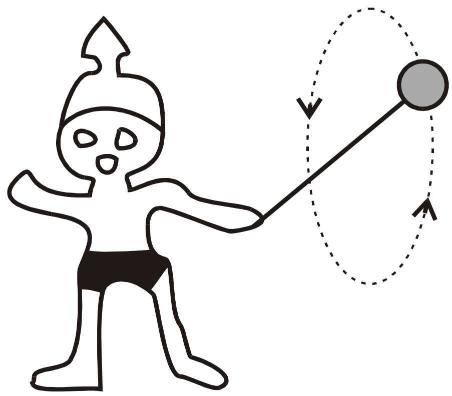
Solution

If in the equation

$$F = \frac{mv^2}{r}$$

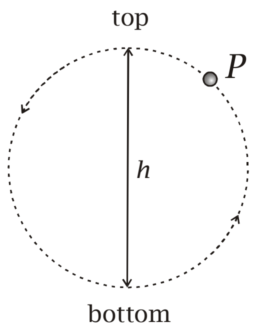
the quantities m and r remain constant, then, as v increases so must F. So a particle can remain in circular motion even if its speed v varies provided the centripetal force varies correspondingly.

Example (3)



A small boy played an amusing game of whirling a ball on a string in the air. The trajectory of the ball described a vertical circle of constant radius. Using energy considerations explain why the speed of the ball could not be as great at the top of the vertical circle as at the bottom. Ignore air resistance.

Solution



As the ball progresses from the bottom of the vertical circle to the top it gains a height h equivalent to the diameter of the circle. Hence it gains gravitational potential energy. Assuming there is no other source of energy being given to the particle (it is not, for instance, supplied with a means of rocket propulsion), this gain of gravitational potential energy can only come from the loss of its kinetic energy. So the particle must be slowing down.

Motion in a vertical circle

When a particle moves in a vertical circle the speed of the particle is constantly changing. This is owing to the transfer of energy that takes place between the particle's gravitational potential energy and its kinetic energy. As the particle ascends it is gaining gravitational potential energy and losing kinetic energy; this is reversed as it descends.

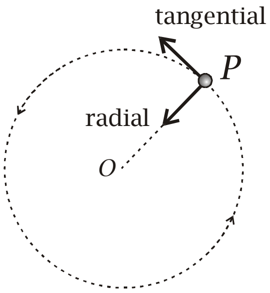
Since the particle is moving in a circle, the radius r is constant. The centripetal force is still given by the equation

$$\text{centripetal force} = \frac{mv^2}{r}$$

but the magnitude of the centripetal force itself varies as the speed of the particle varies.

Tangential and radial components

When a particle moves in a vertical circle there is (1) a centripetal force acting towards the centre of the circle and (2) a force acting in the direction of the motion of the particle that is causing it either to speed up or slow down. Thus, we resolve the forces on the particle into components.



Radial components

These are the components of the motion of P and forces acting on it that are directed towards the centre of the circle of motion – that is, in the diagram above, along the radius OP.

Tangential components

These are the components of the motion of P and forces acting on it that are directed at a tangent to the circle of motion.

If the particle sweeps out an angle θ in t seconds	
Its angular velocity is	$\omega = \frac{d\theta}{dt}$
Its angular acceleration is	$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
The arc length swept out is	$s = r\theta$
Tangential velocity is	$v = \frac{ds}{dt} = r\omega$
Tangential acceleration is	$\frac{dv}{dt} = \frac{d}{dt} r\omega = r\alpha$

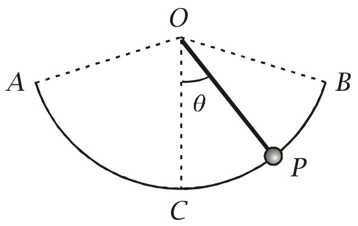
Summary

Angle	**Tangential component**	**Radial component**
angle θ	arc length $s = r\theta$	radius r
angular velocity $\omega = \frac{d\theta}{dt} = \theta$	velocity $v = \frac{ds}{dt} = r\omega$	radial velocity 0
angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$	tangential acceleration $a = \frac{dv}{dt} = r\alpha = r\frac{d\omega}{dt}$	radial acceleration centripetal acceleration $a = r\omega^2 = \frac{v^2}{r}$

The most important piece of information here is the principle that whilst the object remains in motion in a vertical circle, its acceleration in the radial direction is constant and is given by $a = r\omega^2 = \frac{v^2}{r}$. Likewise, the centripetal force remains $F = mr\omega^2 = \frac{mv^2}{r}$.

Example (4)

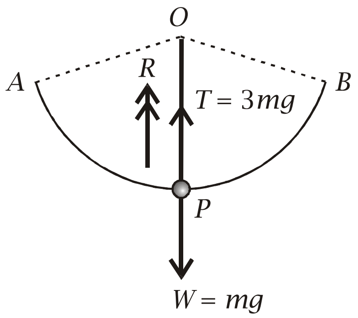
One end of a light rod of length l m is attached to a fixed point O and the other end is attached to a particle P of mass m kg. The particle P is oscillating back and forth along the minor arc AB of a vertical circle with centre O and radius l m, as shown in the diagram.



When P is at its lowest point C its speed is u ms^{-1} and the tension in the rod is $3mg$ N. Show that

$u = \sqrt{2gl}$.

Solution

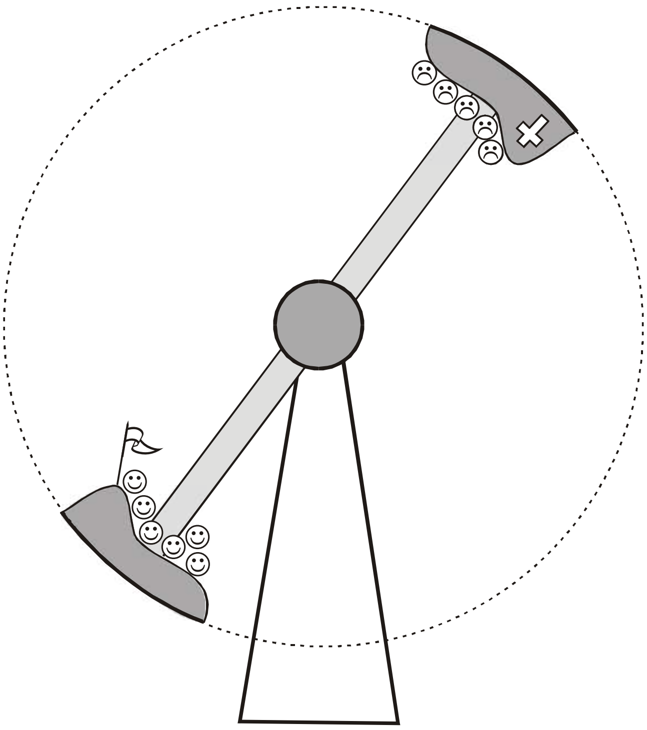


When the particle is at its lowest point it is subject to two forces, its weight $(W = mg)$ acting vertically downwards and the tension in the rod $(T = 3mg)$ acting vertically upwards. The resultant of these two forces supplies the centripetal force $R = \frac{mu^2}{l}$ that keeps the particle in motion in a circle. Hence

$$R = T - W$$
$$\frac{mu^2}{l} = 3mg - mg = 2mg$$
$$u^2 = 2gl$$
$$u = \sqrt{2gl}$$

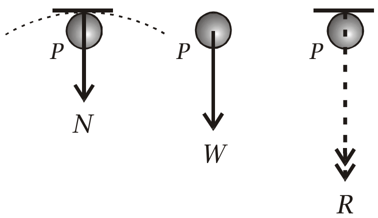
Example (5)

One of the rides at a fairground is contraption that swings children in a vertical circle of radius r, so that at the top of the circle the children are hanging head down.



Find in terms of g and r the minimum speed of the children at the top of the circle in order for the children not to fall out.

Solution



The children may be modelled as a single particle P of mass m kg. There are two forces acting on the children (who in this model are not attached to the car in any way). These are their weight $(W = mg)$ and the normal reaction (contact force, N) between them and the base of the car. This normal reaction is ultimately supplied by the tension in the radial strut connecting the car to the centre of the wheel. The tension pulls the base of the car and the base of the car pushes the children. The children will remain in the car so long as the normal reaction is positive. The children will feel a force acting on then – so they will feel 'stuck' to the seats of the car. The resultant of these two forces is

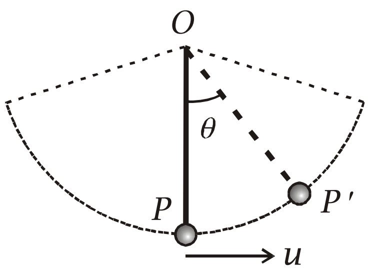
$R = N + W$

This supplies the centripetal force $R = \frac{mv^2}{r}$, which has the effect of keeping the children in vertical circular motion. The children remain in circular motion so long as $N > 0$. That is $\frac{mv^2}{r} > W$. Hence the minimum velocity is given by

$$\frac{mv^2}{r} = W$$
$$\frac{mv^2}{r} = mg$$
$$v = \sqrt{rg}$$

Problem solving by means of conservation of energy

A number of problems involving motion in a vertical circle are conveniently solved by use of the principle of conservation of energy. For example, if a pendulum bob, initially at rest vertically below a fixed point O is given a transverse initial velocity, u, it will begin to move in a vertical circle. The initial velocity imparts to it an initial kinetic energy.



As it moves upwards this kinetic energy is converted to gravitational potential energy. Information about the subsequent motion of the particle can be deduced from the equation

gain in gravitational potential energy = loss of kinetic energy $\quad U = K_E$

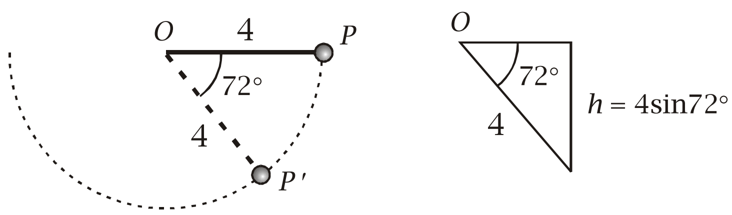
The following example illustrates the application of this principle.

Example (6)

A light inextensible string has length 4 m. One end of this string is attached to a fixed point *O*. To the other end a particle *P* of mass *m* kg is attached. The string is positioned horizontally and tightened. In this position the particle is released from rest. Find the speed of the particle when the angle the string makes with the horizontal is 72°. You may assume no air-resistance.

Solution

To solve this problem we use conservation of energy



As the diagram shows when the angle the string makes with the horizontal is 72°, the particle has fallen a height $h = 4\sin 72°$. The loss of gravitational potential energy that this causes is converted to kinetic energy.

$$K_E = U$$

$$\frac{1}{2}mv^2 = mgh$$

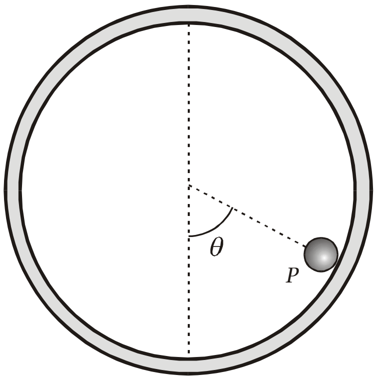
$$\begin{aligned} v &= \sqrt{2gh} \\ &= \sqrt{2 \times 9.81 \times 4\sin 72°} \\ &= 8.64 \text{ ms}^{-1} \quad \text{(3 s.f.)} \end{aligned}$$

Centripetal force as the resultant of weight and other forces

We may be asked to find the magnitude of the contact force between a particle and a surface, or the tension in a string or light rod to which a particle is attached. We have already seen that we may find it by using the principle that so long as the particle remains in vertical circular motion the resultant of the forces supplies the acceleration towards the centre.

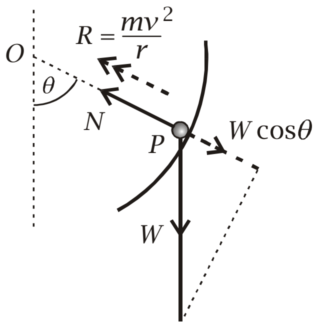
resultant radial force = $R = ma$ $\qquad a = \dfrac{v^2}{r}$

For example, when a particle *P* is in vertical motion within a hollow sphere



The forces acting on the particle P are the weight W and the normal reaction N. The resultant radial force is $R = \frac{mv^2}{r}$. This force is supplied by the normal reaction. Hence, if we know the mass and velocity of the particle we can find the normal reaction. With the angle θ as defined in the diagram, the radial component of the weight is given by

$W_{\text{radial}} = mg\cos\theta$

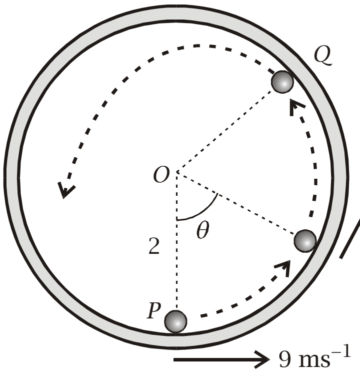


$$\frac{mv^2}{r} = N - mg\cos\theta$$

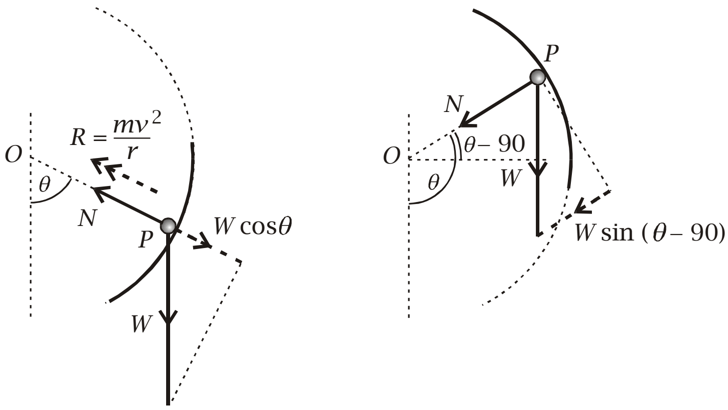
$$N = \frac{mv^2}{r} + mg\cos\theta$$

Example (7)

A body P of mass m kg lies on the inside of a smooth fixed cylinder of radius 2.0 m. It is struck with an initial velocity of $9\ \text{ms}^{-1}$. If the angle made between the line joining P to the centre of the cylinder O and the line descending vertically from O is θ find the value of θ when P leaves the surface of the cylinder.



The diagram illustrates the motion of the mass *P*. It leaves the interior surface of the cylinder at a point *Q*. The forces acting on *P* are its weight and the normal reaction.



Resolving radially

$(\nwarrow)\qquad N = \dfrac{mv^2}{r} + mg\cos\theta$

This formula still applies when $\theta > 90^\circ$, since $\cos\theta$ is negative when $\theta > 90^\circ$. Also when $\theta > 90^\circ$

$(\nwarrow)\qquad N = \dfrac{mv^2}{r} - mg\sin\left(\theta - 90°\right)$

But $\sin\left(\theta - 90°\right) = -\cos\theta$ hence for all values of θ we have $N = \dfrac{mv^2}{r} + mg\cos\theta$.

The particle leaves the surface of the cylinder when $N = 0$.

$$\frac{mv^2}{r} = -mg\cos\theta$$

$$\frac{v^2}{r} = -g\cos\theta$$

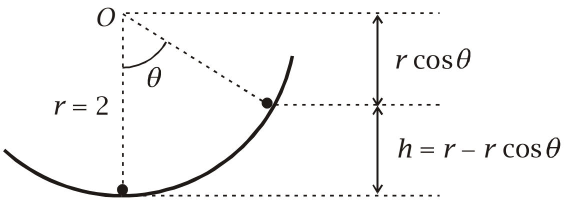
Substituting $g = 9.8,\ r = 2$ into this

$$\frac{v^2}{2} = -9.8\cos\theta$$

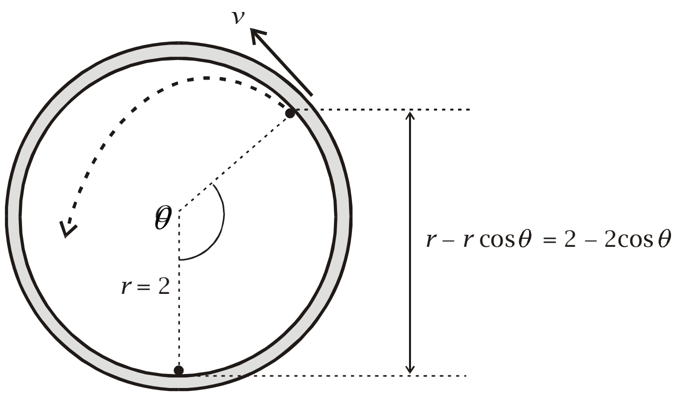
$$v^2 = -19.6\cos\theta$$

We need to find v^2. To do this we use conservation of energy. The total energy of the particle is $\frac{1}{2}mu^2$ at the bottom of the cylinder, where $u = 9$ ms^{-1} is the initial velocity. As the particle climbs up the surface of the cylinder this energy is divided between gravitational potential energy and the remaining kinetic energy of the particle. As the particle moves up the inner surface of the cylinder the height is

$$h = r - r\cos\theta = r(1-\cos\theta)$$



This formula also still applies when $\theta > 90^\circ$, as $\cos\theta$ is negative when $\theta > 90^\circ$ so $1-\cos\theta > 1$.



Total energy = kinetic energy + gravitational potential energy

$$\frac{1}{2}mu^2 = mgh + \frac{1}{2}mv^2$$

$$u^2 = 2gr(1-\cos\theta) + v^2$$

$$v^2 = u^2 - 2gr(1-\cos\theta)$$

Substituting $u = 9,\ g = 9.8,\ r = 2$ into this gives

$v^2 = (9)^2 - 2 \times 9.8 \times 2 \times (1 - \cos\theta)$

$v^2 = 81 - 39.2 + 39.2\cos\theta$

$v^2 = 41.8 + 39.2\cos\theta$

We saw above that when the particle leaves the surface $v^2 = -19.6\cos\theta$. Hence

$-19.6\cos\theta = 41.8 + 39.2\cos\theta$

$$\cos\theta = -\frac{41.8}{(39.2 + 19.6)} = -\frac{41.8}{58.8} = -0.710...$$

$\theta = 135.3° \ (0.1°)$

Once P leaves the surface its motion is described by the motion of a particle under gravity in free fall.